A THEORETICAL INVESTIGATION OF COMBINED FREE- AND FORCED-CONVECTION HEAT-GENERATING LAMINAR FLOW INSIDE VERTICAL PIPES WITH PRESCRIBED WALL TEMPERATURES

by

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B. S., National Taiwan University, China, 1953

A THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE COLLEGE
OF AGRICULTURE AND APPLIED SCIENCE

1959

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INTRODUCTION AND REVIEW OF LITERATURE

The free-convection effect in vertical pipe flow has recently become quite significant in a number of engineering applications. The modifier "vertical" here is employed to mean that the longitudinal dimension of the pipe is parallel to the direction of the generating body-force, such as gravity or centrifugal force. The two important fields where the free-convection phenomenon plays a dominating role nowadays are heat-exchanger problems in nuclear reactors and blade-cooling problems in high speed turbines.

The field of pure free-convection and combined free- and forced-convection pipe flow are, as to be expected, extremely barren compared to that of pure forced-convection. Contrbutions of a theoretical nature are confined to those for parallel-plate channels and circular pipes. Ostrach analyzed pure free-convection, constant heat-generating laminar flow with constant wall temperatures (10); combined free- and forced-convection, constant heat-generating laminar flow in channels with linearly varying wall temperatures (11) and unstable flow in channels with wall temperatures linearly decreasing in the vertical upward direction (12 and 13). Ostrach also pointed out that energy dissipation can be neglected only when a parameter (the product of the thermal coefficient of volumetric expansion, the body force per unit mass, the hydraulic diameter of the pipe cross section and the reciprocal of the specific heat capacity at constant pressure) is small. The effect of energy dissipation has been investigated by successive approximations in all Ostrach's works except one (13). Lietzke (4)

analyzed pure free-convection laminar flow without heat generation in channels with both ends closed, using parameters different from those employed by Ostrach. In Lietzke's analysis, heat fluxes through the walls are prescribed and energy dissipation was neglected entirely.

Lighthill (5) investigated pure free-convection in circular pipes with one or both ends closed. Both laminar and turbulent flows were considered with energy dissipation neglected. Lighthill's analysis is the most inspiring among the only three (2, 5 and 14) that deal with the effects of the presence of the pipe ends. A method of obtaining solutions satisfying the integrated forms of the governing equations, which distinguishes Lighthill's work from the others reviewed here, has been employed by him. Hammitt(2), and Ostrach and Thornton (14) extended further the laminar part of Lighthill's problem by the same integral method.

Hallman (1) treated combined free- and forced-convection, heat-generating laminar flow in circular open-end pipes with linearly varying wall temperature in a way similar to Ostrach's. However, in Hallman's treatment, energy dissipation was neglected and different parameters were used. Furthermore, as quoted in Ostrach (11), approximate solutions of combined free- and forced-convection flow in short pipes and channels were obtained by Martinelli and Boelter. Works by Ostroumov; Hamilton, Poppendiek, Redmond and Palmer; Hamilton, Poppendiek and Palmer; Woodrow; Murgatroyd; Wordsworth; and Timo were also quoted by Ostrach (11 and 12) and Hallman (1).

They all covered the same aspects as Ostrach (10, 11 and 12), Lietzke (4) and Hallman (1). Through private communication, Dr. E. R. G. Eckert informed the author that Lietzke has extended his work on parallel-plate channels to the case of circular annuli but the result has not been made accessible.

Izumi (3) solved unsteady free convection inside vertical pipes by Laplace transform. This work is made known only through Ostrach's review.

The unrestricted way of attacking the problem of the thesis would be to solve simultaneously the continuity, the Navier-Stokes , and the energy equations for a compressible fluid with body force and energy dissipation terms; aided by the equation of state and the laws of variation of the specific heat capacities, thermal conductivity and dynamic viscosity. But all the existing solutions are for semi-incompressible fluids with constant specific heat capacities, thermal conductivity and dynamic viscosity. The fluid is considered to be generally incompressible, but a buoyancy term is introduced by taking into consideration the variation of density with respect to temperature. This trend is also followed in the present investigation. A general theory of the present problem, applied to pipes of any cross section, is developed first. This is essentially a generalization of Ostrach's work on parallel-plate channels. Cases where the axial component of the temperature gradient vanishes are set apart from those where it does not. In all the existing solutions, algebraic elimination of temperature

from the energy and the Navier-Stokes equations is always employed. In the present investigation, however, algebraic elimination of velocity is also used, as sometimes this is more convenient, especially when applying thin-plate analogy. To the best of the author's knowledge, this thin-plate analogy, discovered in the course of the present investigation, has not yet been discussed in the literature. This analogy is not essentially the same as that in pure forced-convection problems (Marco and Han, 6), since, generally, elastically supported plates serve as models here. Other useful analogies are also used.

Following the general theory thus established, the problem is solved for a few specific cross sections. Flow inside circular annuli is treated fully. Solutions for rectagular pipes and pipes of ring-sector cross sections are obtained by using the finite Fourier sine transform. This and some other mathematical means are introduced as formal tools only. Mathematical rigour is not attempted. The above-mentioned specific solutions are the main contributions of the present investigation and, so far, are not available in the literature. Investigations regarding parallel-plate channels and circular pipes have been done rather extensively by others, and hence, are not duplicated here. Some special cases for triangular cross sections are solved by applying thin-plate analogy. Possibility of solution by other analogies or techniques is pointed out. No attempt is made to follow Ostrach in investigating the effects of energy dissipation. The phenomenon of unstable flow is not explored. It is also to be noted that only the problem of prescribed wall

temperature is taken up. The possibility of solving the cases of elliptical pipes and annuli (composed of two confocal ellipses) in terms of Mathieu's functions and other possible directions in continuing the present investigation are pointed out in the SUGGESTIONS.

GENERAL THEORY

General Assumptions

General assumptions mentioned explicitly or implicitly in the INTRODUCTION AND REVIEW OF LITERATURE are summarized here as follows:

1. The pipe wall is cylindrical.

2. The elements of the pipe wall are parallel to the generating body force.

3. The domain investigated is far away from the ends of the pipe.

4. The flow is steady and laminar.

5. The fluid is semi-incompressible; i.e., ρ (see Appendix I for nomenclature) is treated as constant except that the relations

$$\rho = \rho(T)$$

and

$$\beta = -\frac{1}{\rho}\frac{d\rho}{dT}$$

are used when taking the buoyancy effect of density variation into consideration. In practice, suitable values of ρ, β, and/or $\rho\beta$ may be used.

6. c_p, c_v, k and μ of the fluid are constants.

7. Energy dissipation is negligible.

8. $\partial T/\partial z = \text{constant} = A$, i.e., $T = Az + T^*(x_1, x_2)$.

9. The temperature distribution on the wall is known.

Coordinate System and Domain of Variables

A general cylindrical coordinate system is used (Fig. 1). A Cartisian coordinate axis z is chosen parallel to the elements of the pipe wall and negative in the direction of f_z. The mid-point of the length of that portion of the pipe where the flow is well developed is chosen as the origin of z. In any plane perpendicular to the z-axis, a two-dimensional coordinate system (x_1, x_2) is established. Surfaces

$$x_1 = \text{constant}$$

and

$$x_2 = \text{constant}$$

are cylindrical with elements parallel to the z-axis. For convenience, any plane perpendicular to the z-axis will be referred to as the two-dimensional region, as differentiated from the three-dimensional domain consisting of the planes

$$z = \pm 1$$

and the cylindrical pipe-wall surface Γ which is the boundary of the two-dimensional region S.

General Equations

Under the assumptions mentioned above, the continuity equation becomes simply

$$\frac{\partial w}{\partial z} = 0$$

which implies that

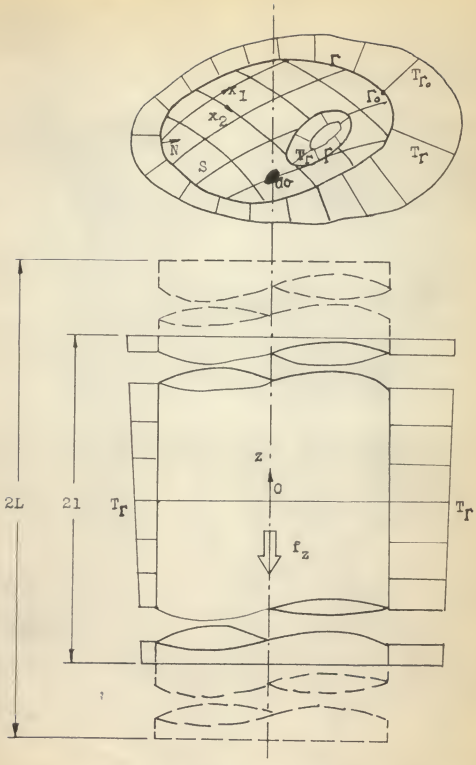

Fig. 1. Coordinate system and domain investigated.

$$w = w(x_1, x_2), \text{ only.}$$

Another useful form of the continuity equation is

$$\rho \iint_S w d\sigma = \text{constant mass through-flow} = g \tag{1}$$

where the mass through-flow g is a known parameter of the problem.

The first two Navier-Stokes equations give

$$(\overleftarrow{\text{grad}}\ p)_{x_1} = 0$$

and

$$(\overleftarrow{\text{grad}}\ p)_{x_2} = 0$$

which imply that

$$p = p(z), \text{ only.}$$

The third Navier-Stokes equation assumes the form

$$\mu \nabla^2 w - \frac{dp}{dz} - \rho f_z = 0. \tag{2}$$

The buoyancy effect can now be brought into equation (2) by adding and subtracting a reference density term:

$$\mu \nabla^2 w - (\rho - \rho_{\Gamma_o}) f_z = \frac{dp}{dz} + \rho_{\Gamma_o} f_z \tag{3}$$

where ρ_{Γ_o} is ρ evaluated along an element Γ_o of Γ. Now assume

$$\rho = \rho(T)$$

and use the definition

$$\beta = -\frac{1}{\rho}\frac{d\rho}{dT}.$$

Three different cases must be considered separately here (Ostrach, 10):

1. If temperature differences in the whole region are small,

ρ and β are essentially constants and suitable average values of them can be used such that

$$\rho\beta = -\frac{\rho-\rho_{\Gamma_0}}{T-T_{\Gamma_0}}$$

or

$$-(\rho-\rho_{\Gamma_0}) = \rho\beta\,(T-T_{\Gamma_0}).$$

2. If temperature differences are large but the fluid is a liquid obeying the relation

$$\rho\beta = \text{constant},$$

then,

$$\frac{d\rho}{dT} = -\rho\beta = \text{constant} = \frac{\rho-\rho_{\Gamma_0}}{T-T_{\Gamma_0}}$$

or

$$-(\rho-\rho_{\Gamma_0}) = \rho\beta\,(T-T_{\Gamma_0}).$$

(If $\rho\beta$ is not exactly a constant, suitable average value of it could be used.)

3. If temperature differences are large but the fluid is a gas obeying the law

$$\rho = \frac{p}{BT}$$

with $p \doteq p_{\Gamma_0}$ (so as to be consistent with the assumption that ρ is a function of T only), then,

$$-(\rho-\rho_{\Gamma_0}) = \frac{p_{\Gamma_0}}{BT_{\Gamma_0}} - \frac{p}{BT}$$

$$= \frac{p}{BT_{\Gamma_0}T}(T-T_{\Gamma_0})$$

$$= \frac{\rho}{T_{\Gamma_0}}(T-T_{\Gamma_0}), \text{ or } \frac{\rho_{\Gamma_0}}{T}(T-T_{\Gamma_0}).$$

But

$$\beta = -\frac{1}{\rho}\frac{d\rho}{dT} = \frac{1}{T},$$

or

$$\beta_{\Gamma_o} = 1/T_{\Gamma_o}.$$

$$\therefore \quad -(\rho - \rho_{\Gamma_o}) = \rho\beta_{\Gamma_o}(T - T_{\Gamma_o}), \text{ or } \rho_{\Gamma_o}\beta(T - T_{\Gamma_o}).$$

Here suitable average values of ρ or β can be used so that

$$-(\rho - \rho_{\Gamma_o}) = \rho\beta(T - T_{\Gamma_o}).$$

For all these cases equation (3) becomes

$$\mu\nabla^2 w + \rho\beta f_z \theta = \frac{dp}{dz} + \rho_{\Gamma_o} f_z \tag{4}$$

provided that suitable average values of ρ, β and/or $\rho\beta$ are selected for each specific case. θ in equation (4) stands for

$$T - T_{\Gamma_o} = (Az + T^*) - (Az + T^*_{\Gamma_o}) = T^* - T^*_{\Gamma_o},$$

and is a function of x_1 and x_2 only. The left-hand side of equation (4) is a function of x_1 and x_2 only while the right-hand side is a function of z only. Therefore, both sides of equation (4) must equal to a constant, say II, and

$$\mu\nabla^2 w + \rho\beta f_z \theta = \text{II}. \tag{5}$$

For an incompressible fluid, the energy equation gives, neglecting energy dissipation,

$$\rho c_v A w = \dot{Q} + k\nabla^2 T \tag{6}$$

[noting that $du = (\partial u/\partial T)_v dT + (\partial u/\partial v)_T dv = c_v dT$, since $dv = 0$] or, since T_{Γ_o} is a function of z only,

$$\rho c_v A w = \dot{Q} + k\nabla^2 \theta. \tag{7}$$

The key equation of the problem can be obtained by eliminat-

ing e from equations (5) and (7):

$$\nabla^4 w + \frac{\rho^2 \beta f_z c_v A}{\mu k} w = \frac{\rho \beta f_z}{\mu k} \dot{Q}. \tag{8}$$

When boundary conditions are specified equation (8) can be solved for w; e can then be obtained from equation (5). The parameter II involved in the solution can be expressed in terms of the known mass through-flow g using equation (1). In this way, it is not necessary to explicitly define the type of flow. Whether a flow is pure free-convection or combined free- and forced-convection is implicitly embodied in the numerical value of g.

The problem can also be solved by eliminating w from equations (5) and (7):

$$\nabla^4 e + \frac{\rho^2 \beta f_z c_v A}{\mu k} e = \frac{\rho c_v A}{\mu k} II - \frac{\nabla^2 \dot{Q}}{k}. \tag{9}$$

After e is obtained from equation (9), equation (7) gives w. The parameter II can again be expressed in terms of g through equation (1).

If it is proved to be more convenient, equatios (8) and (9) may also be paired together to give w and e without resort to equations (5) and (7).

Boundary Conditions

There are two kinds of principal boundary conditions. One of them is the non-slip condition of a viscous fluid at the wall:

$$w = 0 \text{ on } \Gamma \tag{10}$$

or, using equation (7),

$$\nabla^2 e = -\frac{\dot{Q}}{k} \text{ on } \Gamma. \tag{11}$$

The other is the thermal condition at the wall (which, in the present investigation, is the temperature distribution):

$$\begin{aligned} \theta &= \theta_\Gamma \text{ on } \Gamma \\ &= 0 \quad \text{at } \Gamma_o \end{aligned} \tag{12}$$

Using equation (5), equation (12) can be written as

$$\begin{aligned} \nabla^2 w &= \frac{1}{\mu}(\Pi - \rho\beta f_z \theta_\Gamma) \text{ on } \Gamma \\ &= \frac{\Pi}{\mu} \text{ at } \Gamma_o \end{aligned} \tag{13}$$

Beside these two kinds of principal boundary conditions there is a number of auxiliary ones which might be needed at times, e.g., that w and θ are finite; or that

$-\infty < g < \infty$ when both ends are open,

$g=0$ when one or both ends are closed.

When one or both ends are closed there must be pure free-convection in the pipe.

Special Cases with A=0

When A=0 equations (7) and (8) may be written as, respectively

$$k\nabla^2\theta + \dot{Q} = 0 \tag{14}$$

$$\nabla^4 w = \frac{\rho\beta f_z}{\mu k}\dot{Q}. \tag{15}$$

Equations (1), (5), (10), (12) and (13) still hold. Equation (9) becomes the same as equation (14). Equation (11) holds in the whole region S, instead of on Γ only, and thus becomes the same

as equation (14).

Here the problem may be solved by determining θ from equation (14) and then w from equation (5); II is expressed in terms of g by equation (1). It may also be solved by finding w from equation (15) and then θ from equation (5), or w and θ may be solved independently from equations (15) and (14).

Nondimensionalization

In order to nondimensionalize all the equations mentioned above, the hydraulic diameter d of S is used as the characteristic length. For cases with $A \neq 0$, $\sqrt{\frac{c_p K_A Ad}{Pr}}$ and Ad are used as the characteristic velocity and temperature, respectively. For those with A=0, $\sqrt{\frac{c_p K T_{\Gamma_o}}{Pr}}$ and T_{Γ_o} are used. In this manner, dimensionless equations, grouped together for quick reference, are obtained as follows:

<u>$A \neq 0$</u>

$$G_A = \iint_S \omega_A \, d\sigma \tag{16}$$

$$\nabla^4 \omega_A + Ra_A \omega_A = \dot{\theta}_A \tag{17}$$

$$\omega_A = 0 \text{ on } \Gamma \tag{18}$$

$$\left.\begin{aligned} \nabla^2 \omega_A &= F_A - \tau_{A\Gamma} \text{ on } \Gamma \\ &= F_A \quad \text{at } \Gamma_o \end{aligned}\right\} \tag{19}$$

$$\nabla^2 \omega_A + \tau_A = F_A \tag{20}$$

$$\nabla^4\tau_A + Ra_A\tau_A = Ra_A F_A - \nabla^2\dot{z}_A \tag{21}$$

$$\left.\begin{aligned}\tau_A &= \tau_{A\Gamma} \text{ on } \Gamma \\ &= 0 \quad \text{at } \Gamma_o\end{aligned}\right\} \tag{22}$$

$$\nabla^2\tau_A = -\dot{z}_A \text{ on } \Gamma \tag{23}$$

$$Ra_A\omega_A = \dot{z}_A + \nabla^2\tau_A \tag{24}$$

$$\nabla^4\omega_A + Ra_A\omega_A = \dot{z}_A \tag{17}$$

$$\omega_A = 0 \text{ on } \Gamma \tag{18}$$

$$\left.\begin{aligned}\nabla^2\omega_A &= F_A - \tau_{A\Gamma} \text{ on } \Gamma \\ &= F_A \quad \text{at } \Gamma_o\end{aligned}\right\} \tag{19}$$

$$\nabla^4\tau_A + Ra_A\tau_A = Ra_A F_A - \nabla^2\dot{z}_A \tag{21}$$

$$\left.\begin{aligned}\tau_A &= \tau_{A\Gamma} \text{ on } \Gamma \\ &= 0 \quad \text{at } \Gamma_o\end{aligned}\right\} \tag{22}$$

$$\nabla^2\tau_A = -\dot{z}_A \text{ on } \Gamma \tag{23}$$

<u>A=0</u>

$$G = \iint_S \omega\, d\sigma \tag{25}$$

$$\nabla^4\omega = \dot{z} \tag{26}$$

$$\omega = 0 \text{ on } \Gamma \tag{27}$$

$$\left.\begin{aligned}\nabla^2\omega &= F - \tau_\Gamma \text{ on } \Gamma \\ &= F \quad \text{at } \Gamma_o\end{aligned}\right\} \tag{28}$$

$$\nabla^2\omega + \tau = F \tag{29}$$

$$\begin{cases} \nabla^2\tau + \dot{g} = 0 & (30) \\ \left.\begin{array}{l} \tau = \tau_\Gamma \text{ on } \Gamma \\ \quad = 0 \quad \text{at } \Gamma_o \end{array}\right\} & (31) \\ \nabla^2\omega + \tau = F & (29) \\ \omega = 0 \text{ on } \Gamma & (27) \end{cases}$$

$$\begin{cases} \nabla^4\omega = \dot{g} & (26) \\ \omega = 0 \text{ on } \Gamma & (27) \\ \left.\begin{array}{l} \nabla^2\omega = F - \tau_\Gamma \text{ on } \Gamma \\ \quad\quad = F \quad\quad \text{at } \Gamma_o \end{array}\right\} & (28) \\ \nabla^2\tau + \dot{g} = 0 & (30) \\ \left.\begin{array}{l} \tau = \tau_\Gamma \text{ on } \Gamma \\ \quad = 0 \quad \text{at } \Gamma_o \end{array}\right\}. & (31) \end{cases}$$

In rectangular Cartesian coordinate system,

$$d\sigma = d\xi d\eta \tag{32}$$

$$\nabla^2 = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} \tag{33}$$

$$\nabla^4 = \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)^2 . \tag{34}$$

In polar coordinate system,

$$d\sigma = RdRd\phi \tag{35}$$

$$\nabla^2 = \frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2} \tag{36}$$

$$\nabla^4 = \left(\frac{1}{R}\frac{\partial}{\partial R} R \frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\right)^2. \tag{37}$$

Thin-Plate Analogy

An inspection of equation (21) shows that it is analogous to the equation governing small deflection of a thin plate resting on an elastic foundation with τ_A, Ra_A and $Ra_A F_A - \nabla^2 \tau_A$ corresponding to the deflection, the modulus of the foundation divided by the flexural rigidity of the plate, and the lateral load divided by flexural rigidity, respectively (Timoshenko, 17, §47). If the boundary conditions happen to be analogous, too, an exact similarity exists. Similar comments apply to equation (17).

Equation (26) shows an analogy to the equation governing small deflection of a thin plate supported at the edges, without elastic foundation lying beneath it (Timoshenko, 17, §21).

In applying the thin-plate analogy and borrowing known solutions from that field, some generality must be sacrificed, especially when velocity is compared with deflection. In the following, this analogy is introduced only when the more general cases are not attempted. Other analogies are also mentioned at the end of SPECIFIC SOLUTIONS.

Heat Transfer

The Nusselt number used in this investigation is defined as

$$Nu = \frac{hd}{k} = \frac{qd}{\theta^\circ k} = -\frac{d\theta}{dN}d \Big/ \theta^\circ = -\frac{d\tau}{d\nu} \Big/ \tau^\circ, \qquad (38)$$

$$\text{or } -\frac{d\tau_A}{d\nu}\Big/\tau_A^\circ \tag{38}$$

where θ° is a certain reference temperature difference and τ° or τ_A° is its dimensionless counterpart. The selection of θ° is a matter of convenience only, as will be shown later.

SPECIFIC SOLUTIONS

Circular Annuli

In this specific case, Γ is composed of two concentric circles with diameters $D_\lambda=\lambda D_I$ and D_I. Since the hydraulic diameter is $D_\lambda-D_I$, the outer and inner dimensionless radii are

$$\left.\begin{aligned} R_I &= \tfrac{1}{2}\,\frac{1}{\lambda-1} \\ R &= \tfrac{1}{2}\,\frac{\lambda}{\lambda-1} \end{aligned}\right\} , \tag{39}$$

respectively.

Further assumptions are made that T_I^* and T_λ^* are two constants and that $\dot{Q}$ is independent of $\emptyset$. Then the problem becomes axially symmetric, i.e., $\partial/\partial\emptyset=0$ and the following equations may be obtained, T_I being chosen as T_{Γ_o}:

<u>$A\neq 0$</u>

$$G_A=2\pi\int_{R_I}^{R_\lambda}\omega_A R\,dR \tag{40}$$

$$\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)^2\omega_A+Ra_A\omega_A=\dot{z}_A \tag{41}$$

$$\begin{cases} \omega_A = 0 \text{ when } R=R_I \text{ or } R_\lambda & (42) \\ \left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\omega_A = F_A \text{ when } R=R_I & \\ \qquad\qquad\qquad = \epsilon_A F_A \text{ when } R=R_\lambda & (43) \\ \left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right) A + \quad A = F_A & (44) \end{cases}$$

<u>A=0</u>

$$G = 2\pi\int_{R_I}^{R_\lambda} \omega R dR \tag{45}$$

$$\begin{cases} \left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\tau + \dot{z} = 0 & (46) \\ \tau = 0 \text{ when } R=R_I & \\ \quad = (1-\epsilon)F \text{ when } R=R_\lambda & (47) \\ \left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\omega + \tau = F & (48) \\ \omega = 0 \text{ when } R=R_I \text{ or } R_\lambda. & (49) \end{cases}$$

The solutions of equations (40) to (44), when

$$\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)^2 \dot{z}_A = 0, \tag{50}$$

are (Appendix II)

$$\omega_A = \frac{1}{\Delta}\left[\Delta_1 \mathrm{ber}_o(s_A R) + \Delta_2 \mathrm{bei}_o(s_A R) + \Delta_3 \mathrm{ker}_o(s_A R) + \Delta_4 \mathrm{kei}_o(s_A R)\right] + \dot{z}_A / s_A^4 \tag{51}$$

$$\tau_A = F_A + \frac{s_A^2}{\Delta}\left[\Delta_1 \mathrm{bei}_o(s_A R) - \Delta_2 \mathrm{ber}_o(s_A R) + \Delta_3 \mathrm{kei}_o(s_A R) - \Delta_4 \mathrm{ker}_o(s_A R)\right]$$

$$+ \frac{1}{s_A^4}\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\dot{z}_A \tag{52}$$

$$G_A = \frac{2\pi}{s_A \Delta}\left\{\Delta_1\left[R_\lambda \mathrm{bei}_0'(s_A R_\lambda) - R_I \mathrm{bei}_0'(s_A R_I)\right] - \Delta_2\left[R_\lambda \mathrm{ber}_0'(s_A R_\lambda) - R_I \mathrm{ber}_0'(s_A R_I)\right]\right.$$

$$\left. + \Delta_3\left[R_\lambda \mathrm{kei}_0'(s_A R_\lambda) - R_I \mathrm{kei}_0'(s_A R_I)\right] - \Delta_4\left[R_\lambda \mathrm{ker}_0'(s_A R_\lambda) - R_I \mathrm{ker}_0'(s_A R_I)\right]\right\}$$

$$+ \frac{2\pi}{s_A^4}\int_{R_I}^{R_\lambda} \dot{\mathcal{J}}_A R\,dR \tag{53}$$

where

$$\Delta = \begin{vmatrix} \mathrm{ber}_0(s_A R_I) & \mathrm{bei}_0(s_A R_I) & \mathrm{ker}_0(s_A R_I) & \mathrm{kei}_0(s_A R_I) \\ \mathrm{ber}_0(s_A R_\lambda) & \mathrm{bei}_0(s_A R\ ) & \mathrm{ker}_0(s_A R_\lambda) & \mathrm{kei}_0(s_A R_\lambda) \\ \mathrm{bei}_0(s_A R_I) & -\mathrm{ber}_0(s_A R_I) & \mathrm{kei}_0(s_A R_I) & -\mathrm{ker}_0(s_A R_I) \\ \mathrm{bei}_0(s_A R_\lambda) & -\mathrm{ber}_0(s_A R\ ) & \mathrm{kei}_0(s_A R_\lambda) & -\mathrm{ker}_0(s_A R_\lambda) \end{vmatrix} \tag{54}$$

$$\Delta_1 = -\frac{1}{s_A^2}\begin{vmatrix} \dot{\mathcal{J}}_A/s_A^2 & \mathrm{bei}_0(s_A R_I) & \mathrm{ker}_0(s_A R_I) & \mathrm{kei}_0(s_A R_I) \\ \dot{\mathcal{J}}_A/s_A^2 & \mathrm{bei}_0(s_A R_\lambda) & \mathrm{ker}_0(s_A R_\lambda) & \mathrm{kei}_0(s_A R_\lambda) \\ F_A & -\mathrm{ber}_0(s_A R_I) & \mathrm{kei}_0(s_A R_I) & -\mathrm{ker}_0(s_A R_I) \\ \epsilon_A F_A & -\mathrm{ber}_0(s_A R_\lambda) & \mathrm{kei}_0'(s_A R_\lambda) & -\mathrm{ker}_0(s_A R_\lambda) \end{vmatrix} \tag{55}$$

$$\Delta_2 = -\frac{1}{s_A^2}\begin{vmatrix} \mathrm{ber}_0(s_A R_I) & \dot{\mathcal{J}}_A/s_A^2 & \mathrm{ker}_0(s_A R_I) & \mathrm{kei}_0(s_A R_I) \\ \mathrm{ber}_0(s_A R_\lambda) & \dot{\mathcal{J}}_A/s_A^2 & \mathrm{ker}_0(s_A R_\lambda) & \mathrm{kei}_0(s_A R_\lambda) \\ \mathrm{bei}_0(s_A R_I) & F_A & \mathrm{kei}_0(s_A R_I) & -\mathrm{ker}_0(s_A R_I) \\ \mathrm{bei}_0(s_A R_\lambda) & \epsilon_A F_A & \mathrm{kei}_0(s_A R_\lambda) & -\mathrm{ker}_0(s_A R_\lambda) \end{vmatrix} \tag{56}$$

$$\Delta_3 = -\frac{1}{s_A^2}\begin{vmatrix} \mathrm{ber}_0(s_A R_I) & \mathrm{bei}_0(s_A R_I) & \dot{\mathcal{J}}_A/s_A^2 & \mathrm{kei}_0(s_A R_I) \\ \mathrm{ber}_0(s_A R_\lambda) & \mathrm{bei}_0(s_A R_\lambda) & \dot{\mathcal{J}}_A/s_A^2 & \mathrm{kei}_0(s_A R_\lambda) \\ \mathrm{bei}_0(s_A R_I) & -\mathrm{ber}_0(s_A R_I) & F_A & -\mathrm{ker}_0(s_A R_I) \\ \mathrm{bei}_0(s_A R_\lambda) & -\mathrm{ber}_0(s_A R_\lambda) & \epsilon_A F_A & -\mathrm{ker}_0(s_A R_\lambda) \end{vmatrix} \tag{57}$$

$$\Delta_4 = -\frac{1}{s_A^2}\begin{vmatrix} \mathrm{ber}_0(s_A R_I) & \mathrm{bei}_0(s_A R_I) & \mathrm{ker}_0(s_A R_I) & \dot{\mathcal{J}}_A/s_A^2 \\ \mathrm{ber}_0(s_A R_\lambda) & \mathrm{bei}_0(s_A R_\lambda) & \mathrm{ker}_0(s_A R_\lambda) & \dot{\mathcal{J}}_A/s_A^2 \\ \mathrm{bei}_0(s_A R_I) & -\mathrm{ber}_0(s_A R_I) & \mathrm{kei}_0(s_A R_I) & F_A \\ \mathrm{bei}_0(s_A R_\lambda) & -\mathrm{ber}_0(s_A R_\lambda) & \mathrm{kei}_0(s_A R_\lambda) & \epsilon_A F_A \end{vmatrix} \tag{58}$$

Theoretically, F_A in equations (51) to (52) must be expressed in terms of G_A with the aid of equation (53). But it is always easier in practice to work backwards, i.e., to assume a value for F_A and clculate the corresponding G_A.

For the calculation of Nu, Ad is chosen as e , so that

$$\tau_A^{\circ} = 1$$

and $$Nu_{I,\lambda} = -\frac{d\tau_A}{d\nu} = \frac{I}{\mp\lambda}\frac{d\tau_A}{dR}$$

$$= \frac{I}{\mp\lambda}\left\{\frac{s_A^3}{\Delta}\left[\Delta_1 bei_o'(s_A R) - \Delta_2 ber_o'(s_A R) + \Delta_3 kei_o'(s_A R) - \Delta_4 ker_o'(s_A R)\right] + \frac{1}{s_A^4}\left(\frac{d}{dR}\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\dot{z}_A\right\} \quad (59)$$

Both Nu_I and Nu_λ can be calculated from equation (59) using R_I and R_λ, respectively. The negative sign is for the inner wall; the positive for the outer.

In case equation (50) does not hold, the particular integral of equation (41) depends on the form of $\dot{Q}$ and is not simply $\dot{z}_A/s_A^4$. In practice, fortunately, $\dot{Q}$ is either zero or a non-zero constant, so that equations (51) to (53) always apply. For simple forms of $\dot{Q}$ the particular integral may be obtained by standard methods.

It is of interest mathematically, however, to note that equation (41) can be solved without any restriction on $\dot{Q}$ by a special finite Hankel transform developed by Sneddon (16). The result is (Appendix III)

$$\omega_A = 2\sum_m \zeta_m^2 \frac{J_o^2(\zeta_m R_I)\left[J_o(\zeta_m R)G_o(\zeta_m R_\lambda) - J_o(\zeta_m R_\lambda)G_o(\zeta_m R)\right]}{J_o^2(\zeta_m R_\lambda) - J_o^2(\zeta_m R_I)} \cdot \frac{\dot{\xi}_A - \left[\dfrac{J_o(\zeta_m R_\lambda)}{J_o(\zeta_m R_I)} - \epsilon_A\right] F_A}{\zeta_m^4 + s_A^4} \tag{60}$$

where the sum is extended over all the positive roots of the transcendental equation of ζ_m

$$J_o(\zeta_m R_I)G_o(\zeta_m R_\lambda) - J_o(\zeta_m R_\lambda)G_o(\zeta_m R_I) = 0.$$

With equation (60), the whole problem is considered to have been solved mathematically. But the solution in this form is, of course, not easily manageable when numerical results are needed.

For special cases with A=0, equation (46) yields straightforwardly the solution

$$\tau = -\int \frac{1}{R}\left(\int \dot{\xi} R dR\right) dR. \tag{61}$$

The two integration constants are to be evaluated by using equation (47).

In case $\dot{Q}$ is a constant,

$$\tau = -\frac{\dot{\xi} R^2}{4} + C_1 \ln R + C_2.$$

Using equation (47), C_1 and C_2 are determined and

$$\tau = -\frac{\dot{\xi}}{4}(R^2 - R_I^2) + \frac{(1-\epsilon)F + \dfrac{\dot{\xi}}{16}\cdot\dfrac{\lambda+1}{\lambda-1}}{\ln \lambda} \ln(R/R_I). \tag{62}$$

Equation (54) gives, generally,

$$\omega = \int \frac{1}{R}\left\{\int\left[F + \int \frac{1}{R}\left(\int \dot{\xi} R dR\right) dR\right] R dR\right\} dR. \tag{63}$$

Equation (49) can be used to determine the integration constants

involved.

When $\dot{Q}$ is a constant,

$$\omega = \frac{\dot{\mathcal{Q}}}{64}R^4 + \frac{1}{4}\left[F - \frac{\dot{\mathcal{Q}}}{4}R_I^2 + \frac{(1-\epsilon)F + \frac{\dot{\mathcal{Q}}}{16}\frac{\lambda+1}{\lambda-1}}{\ln\lambda}R^2 - \frac{1}{4}\frac{(1-\epsilon)F + \frac{\dot{\mathcal{Q}}}{16}\frac{\lambda+1}{\lambda-1}}{\ln\lambda}R^2\ln(R/R_I)\right.$$

$$+ C_1 \ln R + C_2.$$

C_1 and C_2 can easily be found by using equation (49). The final result is

$$\omega = \frac{\dot{\mathcal{Q}}}{64}(R^4 - R_I^4) + \frac{1}{4}\left[F - \frac{\dot{\mathcal{Q}}}{4}R_I^2 + \frac{(1-\epsilon)F + \frac{\dot{\mathcal{Q}}}{16}\frac{\lambda+1}{\lambda-1}}{\ln}\right](R^2 - R_I^2)$$

$$+ \frac{1}{4}\frac{(1-\epsilon)F + \frac{\dot{\mathcal{Q}}}{16}\frac{\lambda+1}{\lambda-1}}{\ln\lambda}(R_\lambda^2 - R^2)\ln(R/R_I)$$

$$- \frac{1}{\ln\lambda}\left\{\frac{\dot{\mathcal{Q}}}{1024}\frac{\lambda^4-1}{(\lambda-1)^4} + \frac{1}{16}\left[F - \frac{\dot{\mathcal{Q}}}{4}R_I^2 + \frac{(1-\epsilon)F + \frac{\dot{\mathcal{Q}}}{16}\frac{\lambda+1}{\lambda-1}}{\ln\lambda}\right]\frac{\lambda+1}{\lambda-1}\right\}\ln\frac{R}{R_I}. \quad (64)$$

The expression for G when $\dot{Q}$ is constant can be obtained by a straightforward integration of equation (64). The result is too lengthy to quote.

For the calculation of Nu, θ_λ is chosen as θ°, so that

$$\tau^\circ = \tau_\lambda = (1-\epsilon)F$$

and

$$Nu_{I,\lambda} = \frac{I}{\lambda}\frac{1}{(1-\epsilon)F}\frac{d\tau}{dR}.$$

When $\dot{Q}$ is constant,

$$Nu_{I,\lambda} = \frac{I}{\lambda}\frac{1}{(1-\epsilon)F}\left[\frac{\dot{\mathcal{Q}}}{2}R - \frac{(1-\epsilon)F + \frac{\dot{\mathcal{Q}}}{16}\frac{\lambda+1}{\lambda-1}}{\ln\lambda}\frac{1}{R}\right]. \quad (65)$$

Note that when $R_I \to 0$, a circular annulus does not approach a circular pipe, but a pipe with a thin wire along its axis, as a limit.

Rectangular Pipes

In this case, Γ is composed of four straight lines: $x=0$, $x=a$, $y=0$ and $y=b=\alpha a$. Since the hydraulic diameter is $2ab/(a+b)$, the dimensionless side-lengths are

$$\bar{a} = \tfrac{1}{2}\left(\frac{1}{\alpha}+1\right)$$
$$\bar{b} = \tfrac{1}{2}(1+\alpha) \quad . \tag{66}$$

Choosing $T(0,0)$ as T_{Γ_o}, the following equations are obtained:

<u>$A \neq 0$</u>

$$G_A = \int_0^{\bar{b}}\int_0^{\bar{a}} \omega_A \, d\xi \, d\eta \tag{67}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)^2 \omega_A + Ra_A \omega_A = \dot{\omega}_A \tag{68}$$

$$\omega_A = 0 \text{ when } \xi=0,\ \xi=\bar{a},\ \eta=0 \text{ or } \eta=\bar{b} \tag{69}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\omega_A = \begin{cases} F_A - \tau_{A1}(\eta) & \text{when } \xi=0 \\ F_A - \tau_{A3}(\eta) & \text{when } \xi=\bar{a} \\ F_A - \tau_{A2}(\xi) & \text{when } \eta=0 \\ F_A - \tau_{A4}(\xi) & \text{when } \eta=\bar{b} \end{cases} \tag{70}$$

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\omega_A + \tau_A = F_A \tag{71}$$

A=0

$$G=\int_0^{\bar{b}}\int_0^{\bar{a}}\omega\, d\xi\, d\eta \tag{72}$$

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\tau+\dot{g}=0 \tag{73}$$

$$\left.\begin{aligned}\tau&=\tau_1(\eta)\ \text{when}\ \xi=0\\ &=\tau_3(\eta)\ \text{when}\ \xi=\bar{a}\\ &=\tau_2(\xi)\ \text{when}\ \eta=0\\ &=\tau_4(\xi)\ \text{when}\ \eta=\bar{b}\end{aligned}\right\} \tag{74}$$

$$\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\omega+\tau=F \tag{75}$$

$$\omega=0\ \text{when}\ \xi=0,\ \xi=\bar{a},\ \eta=0\ \text{or}\ \eta=\bar{b}. \tag{76}$$

Equation (68), with the boundary conditions (69) and (70), can be solved by double finite Fourier sine transform (Sneddon, 16). The result is (Appendix IV)

$$\omega_A=\frac{4}{\bar{a}\bar{b}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\dot{g}_A-F_A\left[1-(-1)^m\right]\left[1-(-1)^n\right]\left(\frac{m\bar{b}}{n\bar{a}}+\frac{n\bar{a}}{m\bar{b}}\right)}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4}\sin\frac{m\pi\xi}{\bar{a}}\sin\frac{n\pi\eta}{\bar{b}}$$

$$+\frac{4}{\bar{a}\bar{b}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\frac{m\pi}{\bar{a}}\left[\tilde{\tau}_{A1}-(-1)^m\tilde{\tau}_{A3}\right]+\frac{n\pi}{\bar{b}}\left[\tilde{\tau}_{A2}-(-1)^n\tilde{\tau}_{A4}\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4}\sin\frac{m\pi\xi}{\bar{a}}\sin\frac{n\pi\eta}{\bar{b}} \tag{77}$$

The second double series in equation (77) may be simplified to

$$-16F_A \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{1}{mn}\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}}.$$

$$\therefore \omega_A = \frac{4}{\bar{a}\bar{b}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tilde{\zeta}_A}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}}$$

$$-16F_A \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{1}{mn}\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}}$$

$$+\frac{4}{\bar{a}\bar{b}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\frac{m\pi}{\bar{a}}\left[\tilde{\zeta}_{A1}-(-1)^m\tilde{\zeta}_{A3}\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}}$$

$$+\frac{4}{\bar{a}\bar{b}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\frac{n\pi}{\bar{b}}\left[\tilde{\zeta}_{A2}-(-1)^n\tilde{\zeta}_{A4}\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}}. \tag{78}$$

Usually, $T_1=T_2=T_3=T_4=T(0,0)$, i.e., $\tilde{\zeta}_{A1}=\tilde{\zeta}_{A2}=\tilde{\zeta}_{A3}=\tilde{\zeta}_{A4}=0$ so that equation (78) becomes

$$\omega_A = \frac{4}{\bar{a}\bar{b}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\dot{\xi}_A}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}}$$

$$-16F_A \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{1}{mn}\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} . \tag{79}$$

If in addition, $\dot{Q}$ is constant,

$$\omega_A = 16 \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{1}{mn}\left[\frac{\dot{\xi}_A}{\pi^2} - F_A\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} . \tag{80}$$

Equation (80) in turn gives

$$\tau_A = F_A + 16 \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{1}{mn}\left[\dot{\xi}_A\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right) - \pi^2 F_A\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} \tag{81}$$

$$= 16 \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{1}{mn}\left[\dot{\xi}_A\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right) + \frac{s_A^4}{\pi^2 F_A}\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2+s_A^4} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} \tag{81'}$$

$$\left(\text{since } F_A = \frac{16F_A}{\pi^2} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} .\right)$$

$$\text{and } G_A = \frac{64ab}{\pi^2} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{1}{m^2n^2}\left[\frac{\dot{\mathcal{B}}_A}{\pi^2} - F_A\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2 + s_A^4} . \tag{82}$$

For the calculation of Nu, Ad is chosen as 0° so that equations (38) and (81) give

$$Nu_{1,3} = \frac{1}{3}\left(\frac{\partial \tau_A}{\partial \xi}\right)_{1,3} = -16 \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{\pi}{\bar{a}n}\left[\dot{\mathcal{B}}_A\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right) - \pi^2 F_A\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2 + s_A^4} \sin\frac{n\pi\eta}{\bar{b}}$$

$$Nu_{2,4} = \frac{2}{4}\left(\frac{\partial \tau_A}{\partial \eta}\right)_{2,4} = -16 \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{\pi}{m\bar{b}}\left[\dot{\mathcal{B}}_A\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right) - \pi^2 F_A\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2\right]}{\pi^4\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2 + s_A^4} \sin\frac{m\pi\xi}{\bar{a}} .$$

It is to be noted that equation (81') can not be differentiated term by term since, in expanding F_A into its double sine series, a finite jump from $-F_A$ to F_A is introduced at $\xi=0$, $\xi=\bar{a}$, $\eta=0$ and $\eta=\bar{b}$.

Similar techniques can be used to solve equations (73) to (76). The results are as follows:

$$\tau = \frac{4}{\bar{a}\bar{b}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tilde{\dot{\mathcal{B}}} + \frac{m\pi}{\bar{a}}\left[\tilde{\tau}_1 - (-1)^m \tilde{\tau}_3\right] + \frac{n\pi}{\bar{b}}\left[\tilde{\tau}_2 - (-1)^n \tilde{\tau}_4\right]}{\pi^2\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} \tag{84}$$

$$\omega = \frac{4}{\bar{a}\bar{b}} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\tilde{\dot{Q}} + \frac{m\pi}{\bar{a}}\left[\tilde{\tau}_1 - (-1)^m \tilde{\tau}_3\right] + \frac{n\pi}{\bar{b}}\left[\tilde{\tau}_2 - (-1)^n \tilde{\tau}_4\right]}{\pi^4 \left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}}$$

$$- \frac{16}{\pi^4} F \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} . \tag{84'}$$

When $\tau_1 = \tau_2 = \tau_3 = \tau_4 = 0$ and $\dot{Q}$ is a constant,

$$\tau = \frac{16\dot{Q}}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} \tag{85}$$

$$= \frac{16}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{\dot{Q}}{\pi^2} - F\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)}{mn\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} \tag{86}$$

$$G = 64 \frac{\bar{a}\bar{b}}{\pi^6} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\frac{\dot{Q}}{\pi^2} - F\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)}{m^2 n^2 \left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} . \tag{87}$$

All these equations are also derivable from those for $A \neq 0$ by dropping out the subscript A and making s_A equal zero. For the calculation of Nu,

$$\tau\left(\frac{a}{2}, \frac{b}{2}\right) = \frac{16\dot{Q}}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{m-1}{2}} \cdot (-1)^{\frac{n-1}{2}}}{mn\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)}$$

$$= \frac{16\bar{d}}{\pi^4} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{m+n}{2}-1}}{mn\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)}$$

is chosen as τ°, and

$$Nu_{1,3} = -\frac{\pi}{\bar{a}} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\sin\frac{n\pi\eta}{\bar{b}}}{n\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)} \Bigg/ \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{m+n}{2}-1}}{mn\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)} \tag{88}$$

$$Nu_{2,4} = -\frac{\pi}{\bar{b}} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{\sin\frac{m\pi\xi}{\bar{a}}}{m\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)} \Bigg/ \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{(-1)^{\frac{m+n}{2}-1}}{mn\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)} \tag{89}$$

Pipes of Ring-Sector Cross Sections

In this case, Γ is composed of two radial lines, $\phi=0$ and $\phi=\Phi$, and two circular arcs with diameters D_I and $D_\lambda=\lambda D_I$. Since the hydraulic diameter is

$$d = \frac{\frac{\lambda^2-1}{2}\Phi D_I}{\lambda-1+\frac{\lambda+1}{2}\Phi}, \tag{90}$$

the dimensionless radii are

$$\left.\begin{aligned} R_I &= \frac{1}{(\lambda+1)\Phi} + \frac{1}{2}\,\frac{1}{\lambda-1} \\ \text{and} \qquad R_\lambda &= \frac{\lambda}{(\lambda+1)\Phi} + \frac{1}{2}\,\frac{\lambda}{\lambda-1} \end{aligned}\right\} . \tag{91}$$

Note that for a full sector $D_I=0$ and $\lambda=\infty$, while D_I remains finite and equals to D. Hence

$$d = \left.\frac{\frac{1}{2}\Phi\lambda D_I}{\dfrac{1}{1+\frac{1}{\lambda}} + \dfrac{1}{2}\,\dfrac{\Phi}{1-\frac{1}{\lambda}}}\right|_{\substack{\lambda D_I \to D \\ \lambda \to \infty}} = \frac{\frac{1}{2}\Phi D}{1+\frac{1}{2}\Phi} \tag{92}$$

and

$$\left.\begin{aligned} R_I &= 0 \\ R_\lambda &= \frac{1}{\Phi} + \frac{1}{2} \end{aligned}\right\} . \tag{93}$$

When $\Phi=2\pi$ the ring- or full sector becomes an annulus or circle <u>plus</u> one radial line. When Φ is small enough, the sector may be used to approximate a narrow isosceles trapezoid or triangle.

Choosing $T(R_I,0)$ as T_{Γ_0}, the following equations are obtained:

<u>$A\neq 0$</u>

$$G_A = \int_0^{\Phi}\int_{R_I}^{R_\lambda} \omega_A R\,dR\,d\phi \tag{94}$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)^2\omega_A+Ra_A\omega_A=\dot{g}_A \tag{95}$$

$$\omega_A=0 \text{ when } R=R_I,\ R=R_\lambda,\ \phi=0 \text{ or } \phi=\bar{\phi} \tag{96}$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)\omega_A=\begin{cases}F_A-\tau_{AI}(\phi) & \text{when } R=R_I\\ =F_A-\tau_{A\lambda}(\phi) & \text{when } R=R_\lambda\\ =F_A-\tau_{A5}(R) & \text{when } \phi=0\\ =F_A-\tau_{A\bar{\phi}}(R) & \text{when } \phi=\bar{\phi}\end{cases} \tag{97}$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)\omega_A+\tau_A=F_A \tag{98}$$

<u>A=0</u>

$$G=\int_0^{\bar{\phi}}\int_{R_I}^{R_\lambda} R\,dR\,d\phi \tag{99}$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)\tau+\dot{g}=0 \tag{100}$$

$$\tau=\begin{cases}\tau_I(\phi) & \text{when } R=R_I\\ =\tau_\lambda(\phi) & \text{when } R=R_\lambda\\ =\tau_5(R) & \text{when } \phi=0\\ =\tau_{\bar{\phi}}(R) & \text{when } \phi=\bar{\phi}\end{cases} \tag{101}$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)\omega+\tau=F \tag{102}$$

$$\omega=0 \text{ when } R=R_I,\ R=R_\lambda,\ \phi=0 \text{ or } \phi=\bar{\phi}\ . \tag{103}$$

Equation (95), with the boundary conditions (96) and (97), can be solved by finite Fourier sine transform. The result is (Appendix

V)

$$\omega_A = \frac{2}{\Phi} \sum_{n=1}^{\infty} \tilde{\omega}_A \sin \frac{n\pi\phi}{\Phi} , \tag{104}$$

where $\tilde{\omega}_A$ is the solution of

$$\frac{d^4\tilde{\omega}_A}{dR^4} + \frac{2}{R}\frac{d^3\tilde{\omega}_A}{dR^3} - \frac{1}{R^2}\left(1+\frac{2n^2\pi^2}{\Phi^2}\right)\frac{d^2\tilde{\omega}_A}{dR^2} + \frac{1}{R^3}\left(1+\frac{2n^2\pi^2}{\Phi^2}\right)\frac{d\tilde{\omega}_A}{dR}$$

$$+\left[\frac{1}{R^4}\frac{n^2\pi^2}{\Phi^2}\left(\frac{n^2\pi^2}{\Phi^2}-4\right)+Ra_A\right]\tilde{\omega}_A = \tilde{\tilde{\omega}}_A - \frac{1}{R^2}\frac{n\pi}{\Phi}\left\{\left[1-(-1)^n\right]F_A - \tau_{A5} + (-1)^n\tau_{A\Phi}\right\} \tag{105}$$

subject to the boundary conditions

$$\begin{cases} \tilde{\omega}_A = 0 \text{ when } R=R_I \text{ or } R=R_\lambda \\ \dfrac{d^2\tilde{\omega}_A}{dR^2} + \dfrac{1}{R}\dfrac{d\tilde{\omega}_A}{dR} = \dfrac{\Phi}{n\pi}F_A\left[1-(-1)^n\right] - \tau_{AI} \text{ when } R=R_I \\ \qquad\qquad = \dfrac{\Phi}{n\pi}F_A\left[1-(-1)^n\right] - \tau_{A\lambda} \text{ when } R=R_\lambda . \end{cases}$$

For special cases of full sectors, $R_I=0$ and $\tau_{AI}=0$. The boundary conditions for equation (105) become

$$\begin{cases} \tilde{\omega}_A = 0 \text{ when } R\to 0 \text{ or } R=R_\lambda \\ \dfrac{d^2\tilde{\omega}_A}{dR^2} + \dfrac{1}{R}\dfrac{d\tilde{\omega}_A}{dR} = \dfrac{\Phi}{n\pi}F_A\left[1-(-1)^n\right] \text{ when } R\to 0 \\ \qquad\qquad = \dfrac{\Phi}{n\pi}F_A\left[1-(-1)^n\right] - \tilde{\tau}_{A\lambda} \text{ when } R=R_\lambda . \end{cases}$$

It is obvious that contributions to the general solution, which tend to infinity at (0,0), must be ruled out in these special cases.

Equations (98) and (94), then, give

$$\tau_A = F_A - \frac{2}{\Phi}\sum_{n=1}^{\infty}\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR} - \frac{n^2\pi^2}{R^2\Phi^2}\right)\tilde{\omega}_A \sin\frac{n\pi\phi}{\Phi} \tag{106}$$

$$G_A = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n}\left[1-(-1)^n\right]\int_{R_I}^{R_\lambda}\tilde{\omega}_A R\,dR\right\}, \tag{107}$$

respectively. For the calculation of Nu, Ad is chosen as 0°, and

$$Nu_{I,\lambda} = \frac{I}{\lambda}\frac{2}{\Phi}\sum_{n=1}^{\infty}\left(\frac{d}{dR}\frac{1}{R}\frac{d}{dR}R\frac{d}{dR} - \frac{n^2\pi^2}{\Phi^2}\frac{d}{dR}\frac{1}{R^2}\right)\tilde{\omega}_A \sin\frac{n\pi\phi}{\Phi}\Bigg|_{R=R_I,\ R_\lambda} \tag{108}$$

$$Nu_{5,\Phi} = \frac{5}{\Phi}\frac{2}{\Phi^2}\sum_{n=1}^{\infty} n\left(\frac{1}{R^2}\frac{d}{dR}R\frac{d}{dR} - \frac{n^2\pi^2}{R^3\Phi^2}\right)\tilde{\omega}_A \cdot \frac{5}{\Phi}(-1)^n. \tag{109}$$

Similar techniques can be used to solve equations (100) to (103) as follows:

$$\tau = \frac{2}{\Phi}\sum_{n=1}^{\infty}\sin\frac{n\pi\phi}{\Phi} \tag{110}$$

where $\tilde{\tau}$ is the solution of

$$R^2\frac{d^2\tilde{\tau}}{dR^2} + R\frac{d\tilde{\tau}}{dR} - \frac{n^2\pi^2}{\Phi^2} = -R^2\tilde{\tau} - \frac{n\pi}{\Phi}\left[\tau_5 - (-1)^n\tau_{\bar{5}}\right] \tag{111}$$

subject to the boundary conditions

$$\left.\begin{aligned}\tilde{\tau} &= \tilde{\tau}_I \text{ when } R=R_I\\ &= \tilde{\tau}_\lambda \text{ when } R=R_\lambda\end{aligned}\right\}. \tag{112}$$

Equation (111) is a Cauchy's linear equation. Its solution is

$$\tilde{\tau}=C_1R^{\frac{n\pi}{\Phi}}+C_2R^{-\frac{n\pi}{\Phi}}+P_1(R,n) \tag{113}$$

where $P_1(R,n)$ is the particular integral. When τ_5, τ_Φ and $\dot{Q}$ are all constants, the right-hand side of equation (111) becomes

$$-\frac{R^2\dot{Q}\Phi}{n\pi}\left[1-(-1)^n\right]-\frac{n\pi}{\Phi}\left[\tau_5-(-1)^n\tau_\Phi\right]R^2$$

and
$$P_1=-\frac{R^2\dot{Q}\Phi\left[1-(-1)^n\right]}{n\pi(4-n^2\pi^2/\Phi^2)}-\frac{n\pi\left[\tau_5-(-1)^n\tau_\Phi\right]R^2}{\Phi(4-n^2\pi^2/\Phi^2)} \qquad \left(\frac{n\pi}{\Phi}\neq 2\right)$$

$$=-\frac{R^2\ln R}{8}\dot{Q}\left[1-(-1)^n\right]-\frac{R^2\ln R}{2}\left[\tau_5-(-1)^n\tau_\Phi\right] \qquad \left(\frac{n\pi}{\Phi}=2\right),$$

C_1 and C_2 can then be easily determined by using equation (112).

When $R_I=0$, C_2 must be zero to avoid infinite velocity at $R=0$. In case $\tau_\lambda=\tau_5=\tau_\Phi=0$ and $\dot{Q}$=constant,

$$\tilde{\tau}=\frac{\dot{Q}\Phi\left[1-(-1)^n\right]}{n\pi(4-n^2\pi^2/\Phi^2)}\left[(D/2)^2(2R/D)^{\frac{n\pi}{\Phi}}-R^2\right] \qquad \left(\frac{n\pi}{\Phi}\neq 2\right)$$

$$=\frac{\dot{Q}\left[1-(-1)^n\right]}{8}R^2\ln\frac{D}{2R} \qquad \left(\frac{n\pi}{\Phi}=2\right)$$

For the calculation of Nu,

$$\tau\left(\frac{R_I+R_\lambda}{2},\frac{\Phi}{2}\right)=\frac{2}{\Phi}\sum_{n=1,3,5}^{\infty}\tilde{\tau}\left(\text{at } R=\frac{R_I+R_\lambda}{2}\right)\cdot(-1)^{\frac{n-1}{2}}$$

is chosen as τ°. Thus

$$Nu_{I,\lambda} = \frac{I}{\lambda} \sum_{n=1}^{\infty} \frac{d\tilde{\tau}}{dR}\Big|_{R_I,\ R_\lambda} \sin\frac{n\pi\phi}{\Phi} \Big/ \sum_{n=1,3,5}^{\infty} \tilde{\tau}(\text{at } R=\frac{R_I+R_\lambda}{2})\cdot(-1)^{\frac{n-1}{2}} \tag{114}$$

$$Nu_{5,\Phi} = \frac{5}{\Phi} + \frac{\pi}{\Phi} \sum_{n=1}^{\infty} \frac{\tilde{\tau}}{R}_{n} \cdot \frac{5}{\Phi}(-1)^n \Big/ \sum_{n=1,3,5}^{\infty} \tilde{\tau}(\text{at } R=\frac{R_I+R_\lambda}{2})\cdot(-1)^{\frac{n-1}{2}} \tag{115}$$

After $\tilde{\tau}$ is obtained, ω can be found by the same method from equations (102) and (103). The result is

$$\omega = \frac{2}{\Phi} \sum_{n=1}^{\infty} \tilde{\omega} \sin\frac{n\pi\phi}{\Phi} \tag{116}$$

where $\tilde{\omega}$ is the solution of

$$\frac{d^2\tilde{\omega}}{dR^2} + \frac{1}{R}\frac{d\tilde{\omega}}{dR} - \frac{1}{R^2}\frac{n^2\pi^2}{\Phi^2}\tilde{\omega} = \frac{\Phi}{n\pi} F\left[1-(-1)^n\right] - C_1 R^{\frac{n\pi}{\Phi}} - C_2 R^{-\frac{n\pi}{\Phi}} - P_1 \tag{117}$$

subject to the boundary conditions

$$\tilde{\omega}=0 \text{ when } R=R_I \text{ or } R_\lambda\ ; \tag{118}$$

i.e.,

$$\tilde{\omega} = C_3 R^{\frac{n\pi}{\Phi}} + C_4 R^{-\frac{n\pi}{\Phi}} + P_2(R,n) \tag{119}$$

where P_2 is the particular integral. C_3 and C_4 may be determined from equation (118). When τ_5, τ_Φ and $\dot{Q}$ are constants,

$$P_2 = -\frac{C_1}{4(\frac{n\pi}{\Phi}+1)}R^{\frac{n\pi}{\Phi}+2} - \frac{C_2}{4(\frac{n\pi}{\Phi}-1)}R^{-\frac{n\pi}{\Phi}+2} + \frac{\dot{Q}\Phi\left[1-(-1)^n\right]R^4}{n\pi(\frac{n^2\pi^2}{\Phi^2}-4)(\frac{n^2\pi^2}{\Phi^2}-16)}$$

$$-\frac{\left\{\left[1-(-1)^n\right]F-\tau_5+(-1)^n\tau_{\Phi}\right\}R^2}{\frac{n\pi}{\Phi}(\frac{n^2\pi^2}{\Phi^2}-4)} \qquad (\frac{n\pi}{\Phi}\neq 1,\ 2,\ \text{or}\ 4)$$

$$= -\frac{C_1}{8}R^3 - \frac{C_2}{2}R\ln R + \frac{\dot{Q}\left[1-(-1)^n\right]R^4}{45} + \frac{\left\{\left[1-(-1)^n\right]F-\tau_5+(-1)^n\tau_{\Phi}\right\}R^2}{3}$$

$$(\frac{n\pi}{\Phi}=1)$$

$$= -\frac{C_1}{12}R^4 - \frac{C_2}{4} + \frac{\dot{Q}\left[1-(-1)^n\right]R^4\ln R}{96} + \frac{\left\{\left[1-(-1)^n\right]F-\tau_5+(-1)^n\tau_{\Phi}\right\}R^2\ln R}{8}$$

$$(\frac{n\pi}{\Phi}=2)$$

$$= -\frac{C_1}{20}R^6 - \frac{C_2}{12}R^{-2} - \frac{\dot{Q}\left[1-(-1)^n\right]R^4\ln R}{384} - \frac{\left\{\left[1-(-1)^n\right]F-\tau_5+(-1)^n\tau_{\Phi}\right\}R^2}{48}$$

$$(\frac{n\pi}{\Phi}=4).$$

When $R_I=0$, C_2 and C_4 must be zero to avoid infinite temperature and velocity at $R=0$.

Finally,

$$G = \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1}{n}\left[1-(-1)^n\right]\int_{R_I}^{R_\lambda}\tilde{\omega}R\,dR\right\}. \tag{120}$$

Pipes of Other Cross Sections

Method of Thin-Plate Analogy. As pointed out earlier, analogy to the deflection of elastically supported plates exists when $A \neq 0$. Unfortunately, this special field of elasticity is no richer than the field of combined free- and forced-convection pipe flow. New solutions can not be found by observing this analogy. However, a special case of the rectangular pipe solution is compared with its counterpart in thin-plate theory in the following just to show a little detail of the analogy and to check the solution obtained by Fourier transform.

From equation (80),

$$A = \frac{16\dot{g}_A}{\pi^2} \sum_{m=1,3,5}^{\infty} \sum_{n=1,3,5}^{\infty} \frac{1}{mn\left[\pi^4\left(\frac{m^2}{\bar{a}^2} + \frac{n^2}{\bar{b}^2}\right)^2 + s_A^4\right]} \sin\frac{m\pi\xi}{\bar{a}} \sin\frac{n\pi\eta}{\bar{b}} \tag{121}$$

when $F_A=0$, $\dot{Q}$=constant and $\tau_{A1}=\tau_{A2}=\tau_{A3}=\tau_{A4}=0$. In other words, equation (121) is the solution of the equation

$$\begin{cases} \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)^2 \omega_A + Ra_A \omega_A = \dot{g}_A \\ \omega_A = 0 \text{ when } \xi=0,\ \xi=\bar{a},\ \eta=0 \text{ or } \eta=\bar{b} \\ \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right)\omega_A = 0 \text{ when } \xi=0,\ \xi=\bar{a},\ \eta=0 \text{ or } \eta=\bar{b} \end{cases}$$

which is exactly the same as that governing the deflection of an elastically supported rectangular plate under uniform load, with four edges simply and rigidly supported (Timoshenko, 17, §47). A careful comparison checks the correctness of equation (121).

Again, referring back to equations (21) and (23), equation (81'), with $\dot{Q}=0$, is the solution of

$$\begin{cases} \left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)^2\tau_A+Ra_A\tau_A=Ra_AF_A \\ \tau_A=0 \text{ when } \xi=0,\ \xi=\bar{a},\ \eta=0 \text{ or } \eta=\bar{b} \\ \left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\tau_A=0 \text{ when } \xi=0,\ \xi=\bar{a},\ \eta=0 \text{ or } \eta=\bar{b} \end{cases}$$

which is the same as that governing the deflection of an elastically supported, uniformly loaded, rectangular plate with edges simply and rigidly supported. The results in the two fields check. This latter way of applying the analogy is better, since F_A is not necessarily zero, which means a broader scope.

In cases where A=0, thin-plate analogy is more fruitful. For example, pipes of equilateral triangular, or right-angled isosceles triangular cross sections can be treated by borrowing known solutions from thin-plate theory (Timoshenko, 17, §57) as follows.

Equilateral triangle composed of sides

$$\begin{cases} x = -\ a/3 \\ y = \frac{1}{\sqrt{3}}\left(\frac{2}{3}a-x\right) \\ y = \frac{1}{\sqrt{3}}\left(x-\frac{2}{3}a\right) \end{cases}$$

with $\tau_\Gamma=F=0$ and $\dot{Q}$=constant:

$$\omega = \frac{\dot{Q}}{96}\left[\xi^3-3\xi\eta^2-1.5(\xi^2+\eta^2)+0.5\right](1-\xi^2-\eta^2)$$

where d=2a/3.

Right-angled isosceles triangle composed of sides

$$\begin{cases} x = 0 \\ y = 0 \\ y + x = a \end{cases}$$

with $\tau_I = F = 0$ and $\dot{Q}$=constant:

$$\omega = \frac{16\left(1+\frac{\sqrt{2}}{2}\right)^4}{\pi^6}\dot{Q}\left\{\sum_{m=1,3,5}^{\infty}\sum_{n=2,4,6}^{\infty} \frac{n \sin\frac{m\pi\xi}{1+\sqrt{2}/2}\sin\frac{n\pi\eta}{1+\sqrt{2}/2}}{m(n^2-m^2)(m^2+n^2)^2} + \sum_{m=2,4,6}^{\infty}\sum_{n=1,3,5}^{\infty} \frac{m \sin\frac{m\pi\xi}{1+\sqrt{2}/2}\sin\frac{n\pi\eta}{1+\sqrt{2}/2}}{n(m^2-n^2)(m^2+n^2)^2}\right\}$$

where $d=2a/(2+\sqrt{2})$.

Temperature difference and useful parameters can then be obtained as before.

It must be noted that in applying thin-plate analogy, generality of the problem is severely restricted.

<u>Other Analogies or Techniques</u>. For cases where A=0, the problem can be solved by using equations (30), (31), (29) and (27). Owing to the special nature of equations (30) and (29), a number of analogies such as membrane analogy, torsion analogy, various hydrodynamical analogies and heat-conduction analogy exist whenever the condition suits. Method of complex variables is also applicable to these special cases.

SUGGESTIONS

There are many directions in which the present investigation may be extended.

Firstly, of course, numerical calculations including the solution of equation (105) can be performed and experimental work undertaken.

As to the further extension of the theoretical exploration, the author is of the opinion that the cases for the elliptical pipes and annuli (at least when $\dot{Q}=0$) can be solved in terms of Mathieu functions by using elliptical coordinate system. To this end, a treatise (8) and a paper (9) by McLachlan will be helpful. Relaxation and other finite difference methods may also be applied.

In the present investigation wall-temperature distribution is given. The problem may be extended to cover the cases where heat flux at the wall is prescribed. Generally, this boundary condition is more difficult to handle. The three known parameters, $\dot{q}$, $\dot{Q}$ and g, must fulfill an integrated form of energy equation in this type of problem.

Following Ostrach, the effect of friction heating and the problem of unstable flow can be investigated, at least for circular annuli. The numerical calculations involved will make a computer indispensable.

ADDENDUM

Untill after the final copy was prepared in November, 1958, the author was not aware of the following works:

Han, L. S.
Laminar heat transfer in rectangular tubes with combined free and forced-convection. Amer. Soc. Naval Eng., Jour. 67: 163-167, Jan. 1955.

Han L. S.
Laminar heat transfer in rectangular channels. ASME paper No. 58-A-124. to be presented at the annual meeting of the ASME. Nov. 30-Dec. 5, 1958.

In the first one, heat generation was not considered. In the second one the effects of heat generation were evaluated. Both were solved by using the classical Fourier method, i.e., Fourier series with undetermined coefficients. The results checked those obtained in the present investigation by employing the Fourier sine transform.

The first work mentioned above is known to the author only through the second one.

THE AUTHOR.
NOVEMBER 28, 1958.

ACKNOWLEDGMENTS

The author would like to take this opportunity to express his sincere appreciation to Dr. Wilson Tripp whose constant interest and guidance facilitated the execution of the present investigation. Acknowledgments are also due to Professor Linn Helander and Dr. Ralph G. Nevins who helped in many ways.

To his sister, Lan-Chang, and Col. Chuan-Tsi Sun, Ballisticist, Institute of Ordnance Research, Taipei, Taiwan, China, the author is greatly indebted for their long and perservering efforts in encouraging both his undertaking the present investigation and pursuing advanced studies in general.

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Unstable convection in vertical channels with heating from below, including effects of heat sources and friction heating. NACA TN 3458. Washington: 1955.

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APPENDICES

Appendix I

Nomenclature

Dimensions:

$[F]$	Force $(=[ML/t^2])$.
$[L]$	Length.
$[M]$	Mass.
$[Q]$	Thermal energy $(=[FL]=[ML^2/t^2])$.
$[T]$	Temerature.
$[t]$	Time.

Symbols:

A	z-component of the temperature gradient, $[T/L]$.
a	Side length of a rectangular cross-section, $[L]$.
$\bar{a}$	a/d, dimensionless.
B	Gas constant, $[FL/MT]$.
b	Side length of a rectangular cross section, $[L]$.
$\bar{b}$	b/d, dimensionless.
$bei_o(\zeta)$	Bei function of the zeroth order.
$bei_o'(\zeta)$	$d[bei_o(\zeta)]/d\zeta$.
$ber_o(\zeta)$	Ber function of the zeroth order.
$ber_o'(\zeta)$	$d[ber_o(\zeta)]/d\zeta$.
C	A constant.
$C_m^{(n)}$	Constants.
c_p	Specific heat capacity at constant pressure, $[Q/TM]$.
c_v	Specific heat capacity at constant volume, $[Q/TM]$.
D	Diameter of a circle or sector, $[L]$.

d	Hydraulic diameter of the cross section, $[L]$.
F	$\sqrt{\frac{Pr}{c_p K T_{\Gamma_o}}}\,\frac{IId^2}{\mu}$, dimensionless pressure gradient parameter.
F_A	$\sqrt{\frac{Pr \cdot d^3}{c_p K_A A}}\,\frac{II}{\mu}$, dimensionless pressure gradient parameter.
f_z	Body force per unit mass, $[L/t^2]$.
G	$\frac{1}{K}\frac{\beta f_z}{k}g$, dimensionless mass through-flow.
g	Mass through-flow, $[M/t]$.
G_A	$\frac{1}{K_A}\frac{\beta f_z}{k}g$, dimensionless mass through-flow.
$G_o(\zeta)$	$K_o(-i\zeta)$.
Gr	$\frac{\rho^2 \beta f_z T_{\Gamma_o} d^3}{\mu^2}$, Grashof number.
Gr_A	$\frac{\rho^2 \beta f_z A d^4}{\mu^2}$, modified Grashof number.
$\overleftarrow{grad}$	Three-dimensional gradient vector.
h	Heat transfer coefficient, $\dot{q}/\theta^\circ$, $[Q/tL^2T]$.
i	$\sqrt{-1}$.
$I_o(\zeta)$	Modified Bessel function of the first kind, zeroth order.
$J_o(\zeta)$	Bessel function of the first kind, zeroth order.
K	$\gamma Ra(\frac{\beta f_z d}{c_p})$, dimensionless parameter.
k	Thermal conductivity, $[Q/tL^2(T/L)]$.
K_A	$\gamma Ra_A(\frac{\beta f_z d}{c_p})$, dimensionless parameter.

$K_o(\zeta)$	Modified Bessel function of the second kind, zeroth order.
$kei_o(\zeta)$	Kei function of the zeroth order.
$kei_o'(\zeta)$	$d[kei_o(\zeta)]/d\zeta$.
$ker_o(\zeta)$	Ker function of the zeroth order.
$ker_o'(\zeta)$	$d[ker_o(\zeta)]/d\zeta$.
L	Half-length of the pipe, [L].
l	Half-length of that portion of the pipe where the flow is fully developed, [L].
m	1, 2, 3, etc.
N	Inward normal coordinate of Γ, [L].
n	1, 2, 3, etc.
Nu	hd/k, Nusselt number.
p	Pressure, $[F/L^2]$.
P_m	Particular integral of a nonhomogeneous ordinary linear differential equation.
Pr	$\mu c_p/k$, Prandtl number.
$\dot{Q}$	Heat-source term, $[Q/tL^3]$.
$\dot{\mathcal{Q}}$	Qd^2/kT , dimensionless heat-source term.
$\dot{\mathcal{Q}}_A$	Qd/kA, dimensionless heat-source term.
$\dot{q}$	Heat flux at the wall, positive in the direction of N, $[Q/tL^2]$.
R	r/d, dimensionless.
r	Two-dimensional radial coordinate, [L].
Ra	$Pr \cdot Gr/\gamma$, Rayleigh number.
Ra_A	$Pr \cdot Gr_A/\gamma$, modified Rayleigh number.
S	Two-dimensional region investigated.

s_A	$Ra_A^{\frac{1}{4}}$, dimensionless.
T	Temperature, $[T]$.
T^*	$T-Az$, $[T]$.
u	Internal energy per unit mass, $[Q/M]$.
v	Specific volume, $[L^3/M]$.
w	Axial velocity, $[L/t]$.
x	Two-dimensional rectangular Cartesian coordinate, $[L]$.
x_1, x_2	Two-dimensional coordinates, $[L]$ or dimensionless, depending on the specific coordinate system used.
y	Two-dimensional rectangular Cartesian coordinate, $[L]$.
$Y_0(\zeta)$	Bessel function of the second kind (Weber), zeroth order.
z	Cartesian coordinate in the direction of negative f_z, $[L]$.
α	b/a, dimensionless.
β	Thermal coefficient of volumetric expansion, $[1/T]$.
Γ	Boundary of S.
Γ_0	A reference point on Γ.
γ	c_p/c_v, dimensionless.
∇^2	Laplacian, $[1/L^2]$.
$\mathbf{\nabla}^2$	$(\nabla^2)d^2$, dimensionless.
∇^4	$\nabla^2\nabla^2$, $[1/L^4]$.
$\mathbf{\nabla}^4$	$(\nabla^4)d^4$, dimensionless.
ϵ	$1-\tau_\lambda/F$, dimensionless.
ϵ_A	$1-\tau_{A\lambda}/F_A$, dimensionless.
ζ	A parametric variable.
η	y/d, dimensionless.
θ	$T-T_{\Gamma_0}$, $[T]$.

θ°	A certain reference θ, $[T]$.
λ	D_λ/D_I, dimensionless.
μ	Dynamic viscosity, $[M/Lt]$.
ν	N/d, dimensionless.
ξ	x/d, dimensionless.
Π	$dp/dz + \rho_{T_o} f_z$, pressure-grdient parameter, $[F/L^3]$.
ρ	Density, $[M/L^3]$.
σ	Area in the region S, $[L^2]$.
$\bar{\sigma}$	σ/d^2, dimensionless.
τ	θ/T_{T_o}, dimensionless temperature difference.
τ_A	θ/Ad, dimensionless temperature difference.
τ°	θ°/T_{T_o}, dimensionless.
τ_A°	θ°/Ad, dimensionless.
Φ	Angular measurement of a ring- or full sector.
ϕ	Two-dimensional angular coordinate, dimensionless.
$\psi(\ )$	A function of one or several variables.
ω	$\dfrac{w}{\sqrt{\dfrac{c_p K T_{T_o}}{Pr}}}$, dimensionless velocity.
ω_A	$\dfrac{w}{\sqrt{\dfrac{c_p K_A A d}{Pr}}}$, dimensionless velocity.
Δ and Δ_m	Various determinants, dimensionless.

Subscripts:

m	1, 2, 3, etc.
n	1, 2, 3, etc.

I	On the inner wall of a circular annulus or ring-sector.
x_1	x_1-component.
x_2	x_2-component.
Γ	On Γ.
Γ_0	At Γ_0.
λ	On the outer wall of a circular annulus or sector.
$\bar{\phi}$	On the wall $\phi=\bar{\phi}$.
1	On the wall $\xi=0$.
2	On the wall $\eta=0$.
3	On the wall $\xi=\bar{a}$.
4	On the wall $\eta=\bar{b}$.
5	On the wall $\phi=0$.

Applied to dimensional or dimensionless temperatures and temperature differences only.

,	"Or"; e.g., "I,λ"= "on either the inner or the outer wall".

Superscripts:

(n)	', ", etc.
$\sim$	Single finite Fourier sine transform with respect to ξ.
$\sim$	Single finite Fourier sine transform with respect to η, or ϕ.
$\approx$	Double finite Fourier sine transform.
$\smile$	A special finite Hankel transform. (See Appendix III).

Miscellany:

$\overset{I}{\underset{\lambda}{\pm}}$, etc.	"Plus (minus) when on the inner (outer) wall", etc.
$\cdot\,(-1)^n$ (ϕ: 5, 1)	"Times 1, when on $\phi=0$; times $(-1)^n$, when on $\phi=\bar{\phi}$."

Appendix II

Derivation of Equations (51) to (58)

To solve equation (41), which is linear, the complementary function or the solution of the equation (41) with $\dot{z}_A=0$ is found first to be

$$C_1'J_o(i^{\frac{1}{2}}s_AR)+C_2'Y_o(i^{\frac{1}{2}}s_AR)+C_3'I_o(i^{\frac{1}{2}}s_AR)+C_4'K_o(i^{\frac{1}{2}}s_AR)$$

(McLachlan, 7, P.157, Formula 7)

$$=C_1'J_o(i^{\frac{1}{2}}s_AR)+C_2'Y_o(i^{\frac{1}{2}}s_AR)+C_3'\mathrm{ber}_o(s_AR)+iC_3'\mathrm{bei}_o(s_AR)+C_4'\mathrm{ker}_o(s_AR)+iC_4'\mathrm{kei}_o(s_AR)$$

(McLachlan, 7, P.168, Formula 159; P.170, Formula 201).

Now, $$K_o(i^{-\frac{1}{2}}s_AR)=\tfrac{1}{2}i\pi\left[I_o(i^{-\frac{1}{2}}s_AR)+iY_o(i^{\frac{1}{2}}s_AR)\right]$$

(McLachlan, 7, P.164, Formula 110),

$$C_2'Y_o(i^{\frac{1}{2}}s_AR)=iC_2'I_o(i^{-\frac{1}{2}}s_AR)-\frac{2iC_2'}{\pi}K_o(i^{-\frac{1}{2}}s_AR).$$

But $$I_o(i^{-\frac{1}{2}}s_AR)=J_o(i^{\frac{1}{2}}s_AR)$$

(McLachlan,7, P.162, Formula 81),

$$C_1'J_o(i^{\frac{1}{2}}s_AR)+C_2'Y_o(i^{\frac{1}{2}}s_AR)=(C_1'+C_2'i)J_o(i^{\frac{1}{2}}s_AR)-\frac{2iC_2'}{\pi}K_o(i^{-\frac{1}{2}}s_AR)$$

$$=C_1''J_o(i^{\frac{1}{2}}s_AR)+C_2''K_o(i^{-\frac{1}{2}}s_AR).$$

Again $$J_o(i^{\frac{1}{2}}s_AR)=J_o(i^{-3/2}s_AR)=\mathrm{ber}_o(s_AR)-i\,\mathrm{bei}_o(s_AR)$$

(McLachlan, 7, P.162, Formula 81; P.168, Formula 160),

and $$K_o(i^{-\frac{1}{2}}s_AR)=\mathrm{ker}_o(s_AR)-i\,\mathrm{kei}_o(s_AR).$$

Therefore, the complementary function is

$$(C_1''+C_3')\mathrm{ber}_o(s_AR)+i(C_3'-C_1'')\mathrm{bei}_o(s_AR)+(C_4'+C_2'')\mathrm{ker}_o(s_AR)+i(C_4'-C_2'')\mathrm{kei}_o(s_AR)$$

$$=C_1\mathrm{ber}_o(s_AR)+C_2\mathrm{bei}_o(s_AR)+C_3\mathrm{ker}_o(s_AR)+C_4\mathrm{kei}_o(s_AR).$$

When equation (50) holds, it is easily seen that

$$\omega_A = C_1 \mathrm{ber}_o(s_A R) + C_2 \mathrm{bei}_o(s_A R) + C_3 \mathrm{ker}_o(s_A R) + C_4 \mathrm{kei}_o(s_A R) + \dot{z}_A / s_A^4,$$

$$\frac{d}{dR}\omega_A = s_A\left[C_1 \mathrm{ber}_o'(s_A R) + C_2 \mathrm{bei}_o'(s_A R) + C_3 \mathrm{ker}_o'(s_A R) + C_4 \mathrm{kei}_o'(s_A R)\right] + \frac{1}{s_A^4}\frac{d\dot{z}_A}{dR},$$

$$R\frac{d}{dR}\omega_A = s_A\left[C_1 R\mathrm{ber}_o'(s_A R) + C_2 R\mathrm{bei}_o'(s_A R) + C_3 R\mathrm{ker}_o'(s_A R) + C_4 R\mathrm{kei}_o'(s_A R)\right]$$

$$+ \frac{1}{s_A^4} R\frac{d}{dR}\dot{z}_A,$$

$$\frac{d}{dR} R\frac{d}{dR}\omega_A = s_A^2\left[-C_1 R\mathrm{bei}_o(s_A R) + C_2 R\mathrm{ber}_o(s_A R) - C_3 R\mathrm{kei}_o(s_A R) + C_4 R\mathrm{ker}_o(s_A R)\right]$$

$$+ \frac{1}{s_A^4}\frac{d}{dR} R\frac{d}{dR}\dot{z}_A$$

(McLachlan, 7, P.170, Formulas 189 and 190; P.172, Formulas 234 and 235),

$$\left(\frac{1}{R}\frac{d}{dR} R\frac{d}{dR}\right)\omega_A = -s_A^2\left[C_1 \mathrm{bei}_o(s_A R) - C_2 \mathrm{ber}_o(s_A R) + C_3 \mathrm{kei}_o(s_A R) - C_4 \mathrm{ker}_o(s_A R)\right]$$

$$+ \frac{1}{s_A^4}\left(\frac{1}{R}\frac{d}{dR} R\frac{d}{dR}\right)\dot{z}_A.$$

Boundary conditions (42) and (43) now become

$$C_1 \mathrm{ber}_o(s_A R_I) + C_2 \mathrm{bei}_o(s_A R_I) + C_3 \mathrm{ker}_o(s_A R_I) + C_4 \mathrm{kei}_o(s_A R_I) = -\dot{z}_A / s_A^4$$

$$C_1 \mathrm{ber}_o(s_A R_\lambda) + C_2 \mathrm{bei}_o(s_A R_\lambda) + C_3 \mathrm{ker}_o(s_A R_\lambda) + C_4 \mathrm{kei}_o(s_A R_\lambda) = -\dot{z}_A / s_A^4$$

$$C_1 \mathrm{bei}_o(s_A R_I) - C_2 \mathrm{ber}_o(s_A R_I) + C_3 \mathrm{kei}_o(s_A R_I) - C_4 \mathrm{ker}_o(s_A R_I) = -F_A / s_A^4$$

$$C_1 \mathrm{bei}_o(s_A R_\lambda) - C_2 \mathrm{ber}_o(s_A R_\lambda) + C_3 \mathrm{kei}_o(s_A R_\lambda) - C_4 \mathrm{ker}_o(s_A R_\lambda) = -\epsilon_A F_A / s_A^4.$$

C_1, C_2, C_3 and C_4 can then be solved to be

$$C_1 = \Delta_1 / \Delta$$

$$C_2 = \Delta_2 / \Delta$$

$$C_3=\Delta_3/\Delta$$
$$C_4=\Delta_4/\Delta$$

where Δ and Δ_m are determinants defined in equations (54) to (58). Thus, equation (51) is obtained.

Furthermore,

$$\tau_A=F_A-\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\omega_A,$$

and $\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\omega_A$ has already been expressed in terms of bei, ber, kei and ker functions; equation (52) follows directly.

Finally,

$$G_A=2\pi\int_{R_I}^{R_\lambda}\omega_A R dR$$

$$=\frac{2\pi}{\Delta}\int_{R_I}^{R_\lambda}\left[\Delta_1 R\,\mathrm{ber}_o(s_A R)+\Delta_2 R\,\mathrm{bei}_o(s_A R)+\Delta_3 R\,\mathrm{ker}_o(s_A R)+\Delta_4 R\,\mathrm{kei}_o(s_A R)\right]dR$$

$$+\frac{2\pi}{s_A^4}\int_{R_I}^{R_\lambda}\dot{z}_A R dR$$

which leads to equation (53) when integration formulas (McLachlan, 7, P.170, Formulas 189 and 190; P.172, Formulas 234 and 235) are used.

Appendix III

Derivation of Equation (60)

Sneddon (16) has proved that if

$$\bar{\psi}=\int_{R_I}^{R_\lambda} R\psi(R)\left[J_o(\zeta_m R)G_o(\zeta_m R_\lambda)-J_o(\zeta_m R_\lambda)G_o(\zeta_m R)\right]dR$$

in which ζ_m is a root of the equation

$$J_o(\zeta_m R_I)G_o(\zeta_m R_\lambda)-J_o(\zeta_m R_\lambda)G_o(\zeta_m R_I)=0, \tag{A1}$$

the following inversion formula holds true:

$$\psi=2\sum_m \zeta_m \frac{J_o^2(\zeta_m R_I)\bar{\psi}}{J_o^2(\zeta_m R_\lambda)-J_o^2(\zeta_m R_I)}\left[J_o(\zeta_m R)G_o(\zeta_m R_\lambda)-J_o(\zeta_m R_\lambda)G_o(\zeta_m R)\right], \tag{A2}$$

where the summation extends over all the positive roots ζ_m of equation (A1).

Sneddon (16) has also shown that

$$\overline{\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\psi}=\frac{J_o(\zeta_m R_\lambda)}{J_o(\zeta_m R_I)}\psi(R_I)-\psi(R_\lambda)-\zeta_m^2\bar{\psi} \tag{A3}$$

which, when applied twice, yields

$$\overline{\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)^2\psi}=\frac{J_o(\zeta_m R_\lambda)}{J_o(\zeta_m R_I)}\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\psi\Bigg|_{R_I}-\left(\frac{1}{R}\frac{d}{dR}R\frac{d}{dR}\right)\psi\Bigg|_{R_\lambda}$$
$$-\zeta_m^2\left[\frac{J_o(\zeta_m R_\lambda)}{J_o(\zeta_m R_I)}\psi(R_I)-\psi(R_\lambda)-\zeta_m^2\bar{\psi}\right]. \tag{A4}$$

Thus equation (41) can be transformed into

$$\left[\frac{J_o(\zeta_m R)}{J_o(\zeta_m R_I)} - \epsilon_A\right] F_A + \zeta_m^4 \tilde{\omega}_A + s_A \tilde{\omega}_A = \tilde{\xi}_A$$

and

$$\tilde{\omega}_A = \frac{\tilde{\xi}_A - \left[\frac{J_o(\zeta_m R_\lambda)}{J_o(\zeta_m R_I)} - \epsilon_A\right] F_A}{\zeta_m^4 + s_A^4}.$$

Substituting the above expression for $\tilde{\omega}_A$ into equation (A2), equation (60) is established.

Appendix IV

Derivation of Equation (77)

It can be shown (Sneddon, 16) that if

$$\tilde{\psi}(m,\eta)=\int_0^{\bar{a}} \psi(\xi,\eta)\sin\frac{m\pi\xi}{\bar{a}}d\xi ,$$

then,

$$\psi=\frac{2}{\bar{a}}\sum_{m=1}^{\infty}\tilde{\psi}\sin\frac{m\pi\xi}{\bar{a}}.$$

Similarly, if

$$\tilde{\tilde{\psi}}(m,n)=\int_0^{\bar{b}} \tilde{\psi}(m,\eta)\sin\frac{n\pi\eta}{\bar{b}}d\eta ,$$

then,

$$\psi=\frac{4}{\bar{a}\bar{b}}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\tilde{\tilde{\psi}}\sin\frac{m\pi\xi}{\bar{a}}\sin\frac{n\pi\eta}{\bar{b}}. \qquad \text{(A5)}$$

Tables of $\tilde{\psi}$ or $\tilde{\tilde{\psi}}$ for simple functional forms of ψ can be found in Sneddon (16) or Scott (15). Repeated use of the table gives corresponding values of $\tilde{\tilde{\psi}}$. It is only necessary here to mention that, for a constant C, the following finite Fourier sine transforms are listed also in Scott (15) and Sneddon (16):

$$\tilde{C}=\frac{\bar{a}}{m\pi}C\left[1-(-1)^m\right] \qquad \text{(A6)}$$

$$\tilde{\tilde{C}}=\frac{\bar{b}}{n\pi}C\left[1-(-1)^n\right] \qquad \text{(A6')}$$

and
$$\tilde{\tilde{C}} = \frac{\bar{a}\bar{b}}{mn\pi^2} C\left[1-(-1)^m\right]\left[1-(-1)^n\right]. \tag{A7}$$

It is not difficult to show (Sneddon, 16) that

$$\widetilde{\frac{\partial^2\psi}{\partial\xi^2}} = \frac{m\pi}{\bar{a}}\left[\psi(0,\eta)-(-1)^m\psi(\bar{a},\eta)\right] - \frac{m^2\pi^2}{\bar{a}^2}\tilde{\psi}\,. \tag{A8}$$

And it is still easier to see that

$$\widetilde{\frac{\partial^2\psi}{\partial\eta^2}} = \frac{d^2\tilde{\psi}}{d\eta^2}\,. \tag{A9}$$

Therefore,

$$\widetilde{\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi} = \frac{m\pi}{\bar{a}}\left[\psi(0,\eta)-(-1)^m\psi(\bar{a},\eta)\right] - \frac{m^2\pi^2}{\bar{a}^2}\tilde{\psi} + \frac{d^2\tilde{\psi}}{d\eta^2} \tag{A10}$$

which, when applied twice, yields

$$\begin{aligned}\widetilde{\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)^2\psi} = \frac{m\pi}{\bar{a}}&\left\{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(0,\eta)} - (-1)^m\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(\bar{a},\eta)}\right\}\\ &- \frac{m^3\pi^3}{\bar{a}^3}\left[\psi(0,\eta)-(-1)^m\psi(\bar{a},\eta)\right] + \frac{m^4\pi^4}{\bar{a}^4}\tilde{\psi}\\ &-2\frac{m^2\pi^2}{\bar{a}^2}\frac{d^2\tilde{\psi}}{d\eta^2} + \frac{m\pi}{\bar{a}}\frac{d^2}{d\eta^2}\left[\psi(0,\eta)-(-1)^m\psi(\bar{a},\eta)\right] + \frac{d^4\tilde{\psi}}{d\eta^4}\,.\end{aligned} \tag{A11}$$

In case $\psi(0,\eta)=\psi(\bar{a},\eta)=0$, equation (A11) becomes

$$\widetilde{\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)^2\psi} = \frac{m\pi}{\bar{a}}\left\{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(0,\eta)} - (-1)^m\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(\bar{a},\eta)}\right\}$$

$$+\frac{m^4\pi^4}{\bar{a}^4}\tilde{\psi}-2\frac{m^2\pi^2}{\bar{a}^2}\frac{d^2\tilde{\psi}}{d\eta^2}+\frac{d^4\tilde{\psi}}{d\eta^4}. \tag{A12}$$

Analogous to equation (A8), there is the equation

$$\widetilde{\frac{d^2\tilde{\psi}}{d\eta^2}}=\frac{n\pi}{\bar{b}}\left[\tilde{\psi}(m,0)-(-1)^n\tilde{\psi}(m,\bar{b})\right]-\frac{n^2\pi^2}{\bar{b}^2}\tilde{\tilde{\psi}} \tag{A13}$$

which yields, when applied twice,

$$\widetilde{\frac{d^4\tilde{\psi}}{d\eta^4}}=\frac{n\pi}{\bar{b}}\left[\left(\frac{d^2\tilde{\psi}}{d\eta^2}\right)_{(\eta=0)}-(-1)^n\left(\frac{d^2\tilde{\psi}}{d\eta^2}\right)_{(\eta=\bar{b})}\right]$$
$$-\frac{n^3\pi^3}{\bar{b}^3}\left[\tilde{\psi}(m,0)-(-1)^n\tilde{\psi}(m,\bar{b})\right]+\frac{n^4\pi^4}{\bar{b}^4}\tilde{\tilde{\psi}}. \tag{A14}$$

Now, if $\psi(\xi,0)=\psi(\xi,\bar{b})=0$,

$$\tilde{\psi}(m,0)=\left[\int_0^{\bar{a}}\psi(\xi,\eta)\sin\frac{m\pi\xi}{\bar{a}}d\xi\right]_{(\eta=0)}$$

$$=\int_0^{\bar{a}}\psi(\xi,0)\sin\frac{m\pi\xi}{\bar{a}}d\xi \tag{A15'}$$

$$\left.\begin{aligned}&=0\\ \tilde{\psi}(m,\bar{b})&=0\end{aligned}\right\}; \tag{A15}$$

and, therefore,

$$\widetilde{\frac{d^2\tilde{\psi}}{d\eta^2}}=-\frac{n^2\pi^2}{\bar{b}^2}\tilde{\tilde{\psi}} \tag{A16}$$

$$\widetilde{\frac{d^4\tilde{\psi}}{d\eta^4}}=\frac{n\pi}{\bar{b}}\left[\left(\frac{d^2\tilde{\psi}}{d\eta^2}\right)_{(\eta=0)}-(-1)^n\left(\frac{d^2\tilde{\psi}}{d\eta^2}\right)_{(\eta=\bar{b})}\right]+\frac{n^4\pi^4}{\bar{b}^4}\tilde{\tilde{\psi}} \tag{A17}$$

Furthermore, if $\psi(0,\eta)=\psi(\bar{a},\eta)=0$, equation (A10) becomes

$$\widetilde{\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)} = -\frac{m^2\pi^2}{\bar{a}^2}\tilde{\psi}+\frac{d^2\tilde{\psi}}{d\eta^2},$$

which gives, when $\psi(\xi,0)=\psi(\xi,\bar{b})=0$ noting equations (A15) and (A15'),

$$\left(\frac{d^2\tilde{\psi}}{d\eta^2}\right)_{(\eta=0)} = \left[\widetilde{\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)}\right]_{(\eta=0)} = \widetilde{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\right]_{(\eta=0)}}$$

$$\left(\frac{d^2\tilde{\psi}}{d\eta^2}\right)_{(\eta=\bar{b})} = \widetilde{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(\eta=\bar{b})}}.$$

Hence, equation (A17) becomes

$$\widetilde{\frac{d^4\tilde{\psi}}{d\eta^4}} = \frac{n\pi}{\bar{b}}\left\{\widetilde{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(\eta=0)}}-(-1)^n\widetilde{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\right]_{(\eta=\bar{b})}}\right\} + \frac{n^4\pi^4}{\bar{b}^4}\tilde{\tilde{\psi}}. \tag{A18}$$

Using equations (A16) and (A18), equation (A12) is transformed into

$$\widetilde{\widetilde{\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)^2\psi}} = \frac{m\pi}{\bar{a}}\left\{\widetilde{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(\xi=0)}}-(-1)^m\widetilde{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(\xi=\bar{a})}}\right\}$$
$$+\frac{n\pi}{\bar{b}}\left\{\widetilde{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(\eta=0)}}-(-1)^n\widetilde{\left[\left(\frac{\partial^2}{\partial\xi^2}+\frac{\partial^2}{\partial\eta^2}\right)\psi\right]_{(\eta=\bar{b})}}\right\}$$
$$+\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)^2\tilde{\tilde{\psi}}. \tag{A19}$$

Now, $\omega_A(0,\eta)=\omega_A(\bar{a},\eta)=\omega_A(\xi,0)=\omega_A(\xi,\bar{b})=0$. So equation (68) can be transformed by using equations (A19), (70), (A6), (A6') and (A7)

into

$$\left[\pi^4\left(\frac{m^2}{\bar{a}^2}+\frac{n^2}{\bar{b}^2}\right)+s_A^4\right]\tilde{\tilde{\omega}}_A=\tilde{\tilde{q}}_A-F_A\left[1-(-1)^m\right]\left[1-(-1)^n\right]\left(\frac{m\bar{b}}{\bar{a}n}+\frac{\bar{a}n}{m\bar{b}}\right)$$

$$+\frac{m\pi}{\bar{a}}\left[\tilde{\tilde{\tau}}_{A1}-(-1)^m\tilde{\tilde{\tau}}_{A3}\right]+\frac{n\pi}{\bar{b}}\left[\tilde{\tilde{\tau}}_{A2}-(-1)^n\tilde{\tilde{\tau}}_{A4}\right]. \qquad \text{(A20)}$$

Substituting $\tilde{\tilde{\omega}}_A$ obtained from equation (A20) into equation (A5), equation (77) is established.

Similar method can be used to solve equations (73) to (76) yielding equations (84) and (84').

Appendix V

Derivation of Equation (104)

Set
$$\tilde{\psi}=\int_0^{\bar{\Phi}}\psi(R,\phi)\sin\frac{n\pi\phi}{\bar{\Phi}}\,d\phi$$

then, similar to Appendix IV,

$$\widetilde{\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\right)\psi}=\widetilde{\left(\frac{\partial^2}{\partial R^2}+\frac{1}{R}\frac{\partial}{\partial R}+\frac{1}{R}\frac{\partial^2}{\partial\phi^2}\right)\psi}$$
$$=\frac{d^2\tilde{\psi}}{dR^2}+\frac{1}{R}\frac{d\tilde{\psi}}{dR}-\frac{1}{R^2}\frac{n^2\pi^2}{\bar{\Phi}^2}\tilde{\psi}+\frac{1}{R^2}\frac{n\pi}{\bar{\Phi}}\left[\psi(R,0)-(-1)^n\psi(R,\bar{\Phi})\right]$$

and

$$\widetilde{\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\right)^2\psi}=\frac{d^4\tilde{\psi}}{dR^4}+\frac{2}{R}\frac{d^3\tilde{\psi}}{dR^3}-\frac{1}{R^2}\left(1+\frac{2n^2\pi^2}{\bar{\Phi}^2}\right)\frac{d^2\tilde{\psi}}{dR^2}$$
$$+\frac{1}{R^3}\left(1+\frac{2n^2\pi^2}{\bar{\Phi}^2}\right)\frac{d\tilde{\psi}}{dR}+\frac{1}{R^4}\frac{n^2\pi^2}{\bar{\Phi}^2}\left(\frac{n^2\pi^2}{\bar{\Phi}^2}-4\right)\tilde{\psi}$$
$$+\frac{1}{R^2}\frac{n\pi}{\bar{\Phi}}\left[\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\right)\psi\right|_{(R,0)}$$
$$-(-1)^n\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R}\frac{\partial^2}{\partial\phi^2}\right)\psi\Big|_{(R,\bar{\Phi})}$$
$$+\left(\frac{d^2}{dR^2}+\frac{1}{R}\frac{d}{dR}-\frac{1}{R^2}\frac{n^2\pi^2}{\bar{\Phi}^2}\right)\left\{\frac{1}{R^2}\frac{n\pi}{\bar{\Phi}}\left[\psi(R,0)-(-1)^n\psi(R,0)\right]\right\}.$$

Therefore, using equations (96) and (97),

$$\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)^2\omega_A=\frac{d^4\tilde{\omega}_A}{dR^4}+\frac{2}{R}\frac{d^3\tilde{\omega}_A}{dR^3}-\frac{1}{R^2}\left(1+\frac{2n^2\pi^2}{\Phi^2}\right)\frac{d^2\tilde{\omega}_A}{dR^2}$$

$$+\frac{1}{R^3}\left(1+\frac{2n^2\pi^2}{\Phi^2}\right)\frac{d\tilde{\omega}_A}{dR}+\frac{1}{R^4}\frac{n^2\pi^2}{\Phi^2}\left(\frac{n^2\pi^2}{\Phi^2}-4\right)\tilde{\omega}_A$$

$$+\frac{1}{R^2}\frac{n\pi}{\Phi}\left\{\left[1-(-1)^n\right]F_A-\tau_{A5}+(-1)^n\tau_{A\Phi}\right\}$$

and equation (95) can be transformed into

$$\frac{d^4\tilde{\omega}_A}{dR^4}+\frac{2}{R}\frac{d^3\tilde{\omega}_A}{dR^3}-\frac{1}{R^2}\left(1+\frac{2n^2\pi^2}{\Phi^2}\right)\frac{d^2\tilde{\omega}_A}{dR^2}+\frac{1}{R^3}\left(1+\frac{2n^2\pi^2}{\Phi^2}\right)\frac{d\tilde{\omega}_A}{dR}$$

$$+\frac{1}{R^4}\left[\frac{n^2\pi^2}{\Phi^2}\left(\frac{n^2\pi^2}{\Phi^2}-4\right)+Ra_A\right]\tilde{\omega}_A=\dot{g}_A-\frac{1}{R^2}\frac{n\pi}{\Phi}\left\{\left[1-(-1)^n\right]F_A-\tau_{A5}+(-1)^n\tau_{A\Phi}\right\}.$$

This variable-coefficient, linear, non-homogeneous, ordinary differential equation can be solved, subject to the boundary conditions

$$\tilde{\omega}_A=0 \text{ when } R=R_I \text{ or } R$$

$$\left(\frac{1}{R}\frac{\partial}{\partial R}R\frac{\partial}{\partial R}+\frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\right)\omega_A=\frac{d^2\tilde{\omega}_A}{dR^2}+\frac{1}{R}\frac{d\tilde{\omega}_A}{dR}$$

$$=\frac{\Phi}{n\pi}F_A\left[1-(-1)^n\right]-\tilde{\tau}_{AI} \text{ when } R=R_I$$

$$=\frac{\Phi}{n\pi}F_A\left[1-(-1)^n\right]-\tilde{\tau}_{A\lambda} \text{ when } R=R_\lambda.$$

Hence, equation (104) is established.

A THEORETICAL INVESTIGATION OF COMBINED FREE- AND FORCED-CONVECTION HEAT-GENERATING LAMINAR FLOW INSIDE VERTICAL PIPES WITH PRESCRIBED WALL TEMPERATURES

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B. S., National Taiwan University, China, 1953

AN ABSTRACT OF
A THESIS

submitted in partial fulfillment of the

requirements for the degree

MASTER OF SCIENCE

Department of Mechanical Engineering

KANSAS STATE COLLEGE
OF AGRICULTURE AND APPLIED SCIENCE

1959

Recently the free-convection effect in vertical pipe flow has become quite significant in the heat-exchanger problem of reactor engineering and the blade-cooling problem in gas turbine engineering. The modifier "vertical" refers to the general assumption that the longitudinal dimension of the pipe is parallel to the direction of the generating body force such as gravity or centrifugal force.

It is the purpose of the present investigation to find the velocity and temperature distributions inside vertical pipes of various cross sections, through which heat-generating fluids are flowing laminarly. Analytical means were employed throughout. The governing equations of the problem were established first in dimensionless forms subject to nine general assumptions. The most important assumptions are:

1. The fluid is semi-incompressible.

2. Energy dissipation is negligible.

3. The axial component of the temperature gradient is constant.

With the prescribed wall-temperature distribution and the known mass through-flow, the equations were solved for a number of specific cross sections. Nusselt numbers at the wall were also formulated.

Flow inside circular annuli was described in terms of ber, bei, ker and kei functions. The finite Fourier sine transform was employed in obtaining solutions for rectangular pipes and pipes of ring-sector or full sector cross sections. A special finite Hankel transform was used to find, formally, the more general solutions

for circular annuli. These specific solutions are all original contributions of the present investigation.

Also, an analogy to the deflection of elastically supported thin plates was observed for the first time in some special cases.

The possibility of constructing solutions for elliptical pipes and annuli in terms of Mathieu functions was pointed out together with some other possible directions in which the present investigation may be extended.

Most of the mathematical methods used were employed as formal tools only. Unyielding rigour was not attempted. The justification of the results depends generally on the successes of the same methods in other fields of application.

Numerical calculations were not taken up. No verifying experiments were conducted.